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AN INVESTIGATION OF AIRCRAFT HEATERS

XXIX - COMPARISON OF SEVERAL METHODS OF CALCULATING HEAT LOSSES FROM AIRFOILS

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University of California

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SUMMARY

A critical comparison and summary is given of the various methods proposed to date for calculating the unit thermal conductance on the outer surface of a heated wing involving both laminar and turbulent boundary layers, and a new equation is suggested which should indicate the effect of the pressure gradient on the laminar heat transfer to a greater degree than do the expressions presented heretofore. For purposes of comparison the different equations are applied to a Joukowski profile for which the necessary data are accurately known and the results are plotted graphically. The unit thermal conductances in the laminar and turbulent regimes computed by the different methods are found to be in good agreement. A procedure whereby the equations for heat transfer from airfoil surfaces may be applied to a propeller shape is presented by means of an illustrative example.

INTRODUCTION

In the design of heating systems for wings for the purpose of thermal ice prevention a knowledge of the unit thermal conductances on the inner and outer surfaces of the wing is necessary. The internal conductance has probably a somewhat greater effect on the value of the over-all heat transfer but the value of the external conductance is presumably the controlling factor in the case of the distribution of temperature along the wing. The external conductance is a function of many variables depending upon the form of the thermal and fluid boundary layers existing at the airfoil surface. This report will be concerned solely with methods of calculating this latter unit thermal conductance. Excellent summaries of the general problems involved in the design of wing heating systems are given in references 1 to 6.

Five general procedures[1] have been suggested in the literature for computing the heat transfer into laminar boundary layers for incompressible flow along airfoil surfaces and four methods have been proposed for the turbulent regime. These are discussed in turn. In conclusion a new method, which is somewhat more complex than those previously published, is presented for the laminar regime and is intended to represent more nearly the aerodynamic and thermal conditions along a wing section. All the methods discussed treat the laminar and turbulent boundary layers separately, and to date no accurate means of locating the transition point has been evolved. Instead of an accurate specification of this point, current practice favors the assumption that transition occurs at, or near, the point of minimum pressure. This criterion is of course inapplicable for the case of "laminar wings."

The methods of calculation described in this report are each based upon one of four general principles (or simple modifications thereof). These are:

For the laminar regime:

(1) "Reynolds analogy" which states the equivalence of the equations for momentum and heat transfer at Prandtl modulus equal to 1 and in the absence of a pressure gradient.

(2) Pohlhausen's exact solution of the differential equations for heat transfer in a laminar boundary layer along a flat plate for any value of the Prandtl modulus. Also, Colburn's equation for heat transfer along a flat plate based upon Pohlhausen's exact solution.

(3) The postulate that the temperature and velocity distributions in the boundary layer are proportional, the factor of proportionality being calculated by means of a heat balance on the boundary layer (Squire's method).

(4) An integral heat balance over the laminar boundary layer.

For the turbulent regime:

(1) Reynolds analogy

(2) Kármán's modification of Reynolds analogy

(3) Colburn's equation for turbulent heat transfer along a flat plate which is identical with the equations obtained from Reynolds analogy, except that the effect of the variation in the Prandtl modulus from unity is approximately accounted for

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[1]The paper presented by A. N. Tifford (A.S.M.E. Aviation Meeting, Los Angeles, June 1944) was not received before this report was written.

(4) An integral heat balance over the turbulent boundary layer

The methods of analysis summarized in this report also illustrate several approximate procedures for indicating the effect of the pressure gradient existing over an airfoil surface upon the heat transfer into the boundary layer. For purposes of comparison these procedures may be listed under four main headings:

(A) Substitution of the velocity near the airfoil surface (calculated by means of the pressure distributions about the airfoil; see appendix B) into flat-plate equations and the substitution of approximate magnitudes of the local drag coefficient $\left[\text{based on } \left(\frac{\partial u}{\partial y}\right)_{y=0}\right]$ along the airfoil surface into heat-transfer equations derived from a consideration of flow over a flat plate

(B) Substitution of the velocity near the airfoil surface (calculated by means of the pressure distribution about the airfoil, see appendix B) into Colburn's empirical heat-transfer equations for a flat plate

(C) Heat balance on the boundary layer, including the effect of pressure gradient on the velocity distribution with the further postulate that the temperature distribution is proportional to the velocity distribution

(D) Heat balance on the boundary layer solving the heat transfer and hydrodynamic equations simultaneously

With these classifications the methods described herein may be tabulated as follows:

General principle / Method	LAMINAR: Reynolds analogy	LAMINAR: Pohlhausen flat-plate solution: Analytical	LAMINAR: Pohlhausen flat-plate solution: Colburn equation	LAMINAR: Squire's method	LAMINAR: Heat balance	TURBULENT: Reynolds analogy	TURBULENT: Von Kármán's modification of Reynolds analogy	TURBULENT: Colburn equation	TURBULENT: Heat balance
Allen and Look	A								
Frick and McCullough		A				A			
Martinelli and others			B					B	
Squire				C			A		C
This report					D				

The material in this report is divided into two sections; the first section describes the methods for calculations in the laminar regime and the second section describes the methods for the turbulent regime.

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SYMBOLS

A	area of airfoil equal to chord times span, square feet
a	thermal diffusivity of fluid $\left(\frac{k}{3600C_p\gamma}\right)$, (sq ft)/(sec)
C_f	drag coefficient $\left(F_D = \frac{1}{2}C_f\rho A u_\infty^2\right)$
C_L	lift coefficient $\left(F_L = \frac{1}{2}C_L\rho A u_\infty^2\right)$
C_p	heat capacity of fluid at constant pressure, Btu/(lb)(°F)
c	wing chord, feet
d	diameter of equivalent cylinder, feet
f_c	average unit thermal convective conductance, for length L, between airfoil surface and air, Btu/(hr)(sq ft)(°F)
f_{c_x}	local or point unit thermal convective conductance between airfoil surface and air at any point x, Btu/(hr)(sq ft)(°F)
f_{c_φ}	local or point unit thermal convective conductance between airfoil surface and air at any angle φ, Btu/(hr)(sq ft)(°F)
F_D	drag force, pounds
F_L	lift force, pounds
J	mechanical equivalent of heat (778), (ft-lb)/Btu
k	thermal conductivity of fluid, Btu/(hr)(sq ft)(°F/ft)
Nu	Nusselt modulus
p	pressure, (lb)/(sq ft)
q_x	heat transferred at any point x, Btu/(hr)
R	ideal gas constant, (ft-lb)/(lb)(°R)
Re_c	Reynolds modulus based on wing chord $\left(\frac{u_\infty c}{\nu}\right)$
Re_D	Reynolds modulus based on cylinder diameter $\left(\frac{u_\infty d}{\nu}\right)$
Re_x	Reynolds modulus based on length x $\left(\frac{u_\infty x}{\nu}\right)$

T temperature of fluid in boundary layer, $^{\circ}R$

T_1 temperature of fluid at edge of boundary layer, $^{\circ}R$

T_∞ temperature of fluid in free stream, $^{\circ}R$

T_0 temperature of surface, $^{\circ}R$

U velocity of fluid near airfoil surface in x-direction calculated from Bernoulli's equation, $U\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$, (ft)/(sec)

u velocity of fluid in boundary layer in x-direction, (ft)/(sec)

u_∞ velocity of fluid in free stream in x-direction, (ft)/(sec)

v velocity of fluid in boundary layer in y-direction, (ft)/(sec)

x length along airfoil profile measured from point of stagnation, feet

y distance normal to airfoil surface, feet

α angle of attack of airfoil, degree

λ dimensionless parameter in Pohlhausen solution $\left(\frac{\delta^2}{\nu}\frac{dU}{dx}\right)$

δ thickness of laminar boundary layer, feet

δ_H thickness of thermal boundary layer, feet

δ_1 displacement thickness of hydrodynamic boundary layer $\left(\int_0^\infty \left(1 - \frac{u}{U}\right) dy\right)$, feet

δ_2 displacement thickness of thermal boundary layer $\left(\int_0^\infty \left(\frac{T - T_\infty}{T_0 - T_\infty}\right) dy\right)$, feet

δ_T "characteristic length" in turbulent boundary layer, feet

φ angle between radius through point on cylinder and radius through point of stagnation measured at axis of cylinder, degree; also functions defined by equations (40) and (47)

μ	absolute viscosity of fluid, (lb)(sec)/(sq ft)
ν	kinematic viscosity of fluid, (sq ft)/(sec)
γ	weight density of fluid, (lb)/(ft3)
ρ	mass density of fluid (γ/g), (lb)(sec^2)/(ft^4)
θ	momentum thickness of boundary layer $\left(\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right)$, feet
ζ	a dimensionless quantity $\left(\sqrt{\frac{\rho U^2}{\tau_o}}\right)$
ϵ	"eddy viscosity" defined by the equation $\frac{\tau}{\rho} = \epsilon \frac{du}{dy}$, (lb)(sec)/(sq ft)
τ_o	drag at the surface, (lb)/(sq ft)
Pr	Prandtl modulus $\left(\frac{3600\mu C_p g}{k}\right)$

DISCUSSION OF METHODS

Laminar Regime

The first part of this section consists of a generalized discussion of the theory for the calculation of heat flow into laminar boundary layers. The details of five methods are then stated at the end of the section.

(1) Reynolds analogy.— With the usual postulates of boundary-layer theory (see appendix B), the hydrodynamic equation for steady two-dimensional flow in a laminar boundary layer in the absence of a pressure gradient[2] is (reference 7, vol. I, ch. IV)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

The equation for the temperature distribution in such a boundary layer is (reference 7, vol. II, p. 610)

[2] A flat plate oriented in the direction in which the fluid is flowing will satisfy the requirements of equation (1). (See appendix B.)

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{Pr} \frac{\partial^2 T}{\partial y^2} \tag{2}$$

When the velocity and temperature of equations (1) and (2) are expressed in dimensionless form by dividing, respectively, by u_∞ the velocity of the free stream (in the case of a flat plate the velocity in the free stream is equal to the velocity at the edge of the boundary layer) and by $\Delta T = T_0 - T_\infty$, the difference in temperature between the surface and the free stream, there is obtained

$$u \frac{\partial\left(\frac{u}{u_\infty}\right)}{\partial x} + v \frac{\partial\left(\frac{u}{u_\infty}\right)}{\partial y} = \nu \frac{\partial^2\left(\frac{u}{u_\infty}\right)}{\partial y^2} \tag{3}$$

and

$$u \frac{\partial\left(\frac{T}{\Delta T}\right)}{\partial x} + v \frac{\partial\left(\frac{T}{\Delta T}\right)}{\partial y} = \frac{\nu}{Pr} \frac{\partial^2\left(\frac{T}{\Delta T}\right)}{\partial y^2} \tag{4}$$

Inspection of equations (3) and (4) reveals that if $Pr = 1$ and if both equations have the same boundary conditions, the solutions of the two equations are identical $\left(\text{that is, } \frac{u}{u_\infty} = \frac{T}{\Delta T}\right)$ and thus the temperature and velocity distributions are exactly similar.

If the solutions are identical, then

$$\left[\frac{\partial\left(\frac{u}{u_\infty}\right)}{\partial y}\right]_{y=0} = \left[\frac{\partial\left(\frac{T}{\Delta T}\right)}{\partial y}\right]_{y=0} \tag{5}$$

or

$$\frac{\left(\frac{\partial T}{\partial y}\right)_{y=0}}{\left(\frac{\partial u}{\partial y}\right)_{y=0}} = \frac{\Delta T}{u_\infty} = \frac{T_0 - T_\infty}{u_\infty} \tag{6}$$

This fundamental relation expresses the well-known Reynolds analogy which states that there exists a direct relation between the temperature and velocity gradients, which respectively control the rate of heat transfer and the drag (shear) at each point on the surface of a flat plate losing heat to a stream of fluid.

Since

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{C_{f_x}\rho u_\infty^{\ 2}}{2} \tag{7}$$

and

$$f_{c_x} = \frac{k}{T_0 - T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{7a}$$

equation (6) may be written as

$$\frac{f_{c_x}}{\gamma C_p u_\infty} Pr = \frac{C_{f_x}}{2} \tag{8}$$

Since in the derivation of equation (8), Pr was postulated to be equal to unity, the equation should be written as

$$\frac{f_{c_x}}{\gamma C_p u_\infty} = \frac{C_{f_x}}{2} \tag{9}$$

It cannot be overemphasized that equations (6) and (9) are valid only if each of the two fundamental conditions upon which they are based is satisfied, namely that $\frac{\partial p}{\partial x} = 0$, and $Pr = 1$. If the pressure gradient along the surface is not zero, equation (1) no longer describes the velocity in the boundary layer but a term involving the pressure gradient must be added. That is, for laminar boundary-layer flow over a surface other than a flat plate, the hydrodynamic equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{10}$$

instead of equation (1), and the dimensionless equations for the velocity and temperature will no longer be identical. Reference to equations (3) and (4) indicates clearly also that unless $Pr = 1$, the equations will no longer be identical.

A simultaneous solution of equations (10) and (2) is difficult except for certain special variations of $\frac{\partial p}{\partial x}$ with x (reference 7, vol. II, pp. 631–635)(appendix B). These special cases indicate, however, that an approximate correction for the pressure gradient existing along the airfoil surface may be made by

(a) Substituting U, the velocity near the airfoil surface (calculated from the pressure distribution about the airfoil) for u_∞ in the Reynolds analogy

(b) Substituting the local drag coefficient along the airfoil (based on $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ along the airfoil) for the flat-plate drag coefficient required by the Reynolds analogy

Thus the Reynolds analogy modified approximately to account for pressure gradient is

$$\frac{f_{c_x}}{\gamma c_p U} = \frac{c_{f_x}}{2} \tag{11}$$

(2) Pohlhausen solution.- Pohlhausen (reference 8) has solved equations (1) and (2) simultaneously for magnitudes of Pr other than unity by substituting the Blasius series solution for the velocity in equation (2) to obtain the temperature distribution in a laminar boundary layer along a flat plate. His solution may be written

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0.332 \sqrt[3]{Pr} \sqrt{\frac{u_\infty}{\nu x}} \left(T_o - T_\infty\right) \tag{12}$$

Since the Blasius solution for the velocity distribution in a laminar boundary layer along a flat plate is (reference 7, vol. I, p. 135)

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.332 \sqrt{\frac{u_\infty}{\nu x}}\, u_\infty \tag{13}$$

the ratio of the temperature gradient at the wall to the velocity gradient now becomes

$$\frac{\left(\frac{\partial T}{\partial y}\right)_{y=0}}{\left(\frac{\partial u}{\partial y}\right)_{y=0}} = Pr^{1/3} \frac{\left(T_o - T_\infty\right)}{u_\infty} \tag{14}$$

When this expression is compared with equation (6) it is noted that the Pohlhausen solution yields the same results as the Reynolds analogy except for the inclusion of the term involving the Prandtl modulus. Thus, equation (14) may be utilized to calculate heat losses from a flat plate for magnitudes of the Prandtl modulus differing from unity. The condition that the pressure gradient along the plate is zero must still be fulfilled however, and equation (14) is again strictly valid only for a flat plate.

As before, equation (14) may be rearranged, for

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{C_{f_x}\rho u_\infty^{\ 2}}{2}$$

and

$$f_{c_x} = \frac{k}{T_0 - T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Thus by substitution into equation (14)

$$\frac{f_{c_x}}{3600\gamma C_p u_\infty} \Pr^{2/3} = \frac{C_{f_x}}{2} \tag{15}$$

Colburn has demonstrated that equation (15) satisfactorily correlates experimental data for the heat loss from a flat plate.

By the use of the equation for the local drag coefficient along a flat plate (reference 7, vol. I, p. 135)

$$C_{f_x} = 0.332 \mathrm{Re}_x^{-0.5} \tag{16}$$

Equation (15) may be rewritten as

$$\frac{f_{c_x}}{3600\gamma C_p u_\infty} \Pr^{2/3} = 0.332 \mathrm{Re}_x^{-0.5} \tag{17}$$

where

$$\mathrm{Re}_x = \frac{u_\infty \gamma x}{\mu g}$$

Equation (15) may be modified as before for application to airfoils by the substitution of U for u_∞ and the calculation of the local drag coefficient for the airfoil. Thus equation (15) becomes

$$\frac{f_{c_x}}{3600\gamma C_p U} \Pr^{2/3} = \frac{C_{f_x}}{2} \tag{18}$$

Equation (17) may also be modified for application to airfoils by the substitution of U for u_∞. The correction for differences in the local drag between the airfoil and the flat plate is not accounted for, however. Thus

$$\frac{f_{c_x}}{3600\gamma C_p U} Pr^{2/3} = 0.332 Re_x^{-0.5} \tag{19}$$

Equation (19) is termed the Colburn laminar equation in this report.

<u>(3) Proportionality between velocity and temperature distributions.</u>— If it is postulated that the temperature distribution is everywhere proportional to the velocity distribution, then

$$\frac{T - T_\infty}{T_o - T_\infty} = 1 - \frac{u}{U}$$

If, in addition, the ratio $\frac{u}{U}$ is expressed as a known function of the parameter $\frac{y}{\delta_1}$ where δ_1 is the hydrodynamic-boundary-layer thickness, that is,

$$\frac{u}{U} = f\left(\frac{y}{\delta_1}\right)$$

then from the similarity of the temperature and velocity profiles

$$\frac{T - T_\infty}{T_o - T_\infty} = 1 - f\left(\frac{y}{\delta_2}\right)$$

where f is the same function as in the hydrodynamic case and δ_2 is the thickness of the thermal boundary layer. The problem of determining the temperature distribution then reduces to one of expressing δ_2 in terms of the known value of δ_1. The ratio $\frac{\delta_2}{\delta_1}$ is in general a function of both the Prandtl modulus and the pressure distribution and reduces to unity only in the case of heat transfer from a flat plate at $Pr = 1$. Squire's contribution consists in deriving this functional relation by means of the energy balance described in the following section.

<u>(4) Heat balance.</u>— When an incompressible fluid is considered and the "dissipation" effect is neglected, an energy balance may be made across a section of the boundary layer of width dx at a point x, (see fig. 1) as follows. The amount of excess heat being carried across a normal to the surface at x is $C_p\gamma \int_0^\infty u(T - T_\infty)\, dy$ and the difference

between the value of this expression at $x + dx$ and the value at x must be equal to the amount of heat entering the fluid from the surface, that is, $-k\left(\frac{\partial T}{\partial y}\right)_{y=0}$. For uniform density and zero dissipation, the energy equation for the thermal layer is

$$\frac{\partial}{\partial x}\left[\int_0^\infty u(T - T_\infty)\ dy\right] = \frac{k}{C_p \gamma}\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{18a}$$

If u and T are known, the point unit conductance along any surface as a function of y is then given by

$$\frac{f_{c_x}}{3600\gamma C_p U} = \frac{1}{U}\frac{\partial}{\partial x}\left[U \int_0^\infty \frac{u}{U}\left(\frac{T - T_\infty}{T_o - T_\infty}\right) dy\right] \tag{19a}$$

It should be noted that, in general, principles (1) and (2) are based on a knowledge of the velocity gradient at the surface of the body, whereas principle (3) requires that the velocity gradient at the wall as well as the velocity distribution be known. Principle (4) requires the knowledge of both the temperature and velocity distributions in the laminar boundary layer but does not require an exact knowledge of the velocity and temperature gradients at the wall.

In the following paragraphs several methods which have been proposed for the calculation of laminar-boundary-layer unit conductance along wings are compared.

Method of Allen and Look.- Allen and Look (reference 9) treat only the case of heat transfer into laminar boundary layers, basing the derivation upon Reynolds analogy as expressed by equation (11),

$$\frac{f_{c_x}}{3600\gamma C_p U} = \frac{C_{f_x}}{2}$$

or

$$f_{c_x} = \frac{1}{2}C_{f_x}\gamma C_p U \times 3600$$

In order to evaluate the local drag coefficient C_{f_x}, the Blasius type velocity profile, which represents the solution of the differential equation for flow in an incompressible boundary layer along a flat plate (with zero pressure gradient), is postulated by the authors to apply over the entire airfoil. In terms of the laminar-boundary-layer[3] thickness δ, the local drag coefficient along a flat plate becomes

$$c_{f_x} = \frac{2(0.765)\mu g}{\gamma U\delta} \tag{20}$$

When equation (20) is substituted in equation (11) and rearranged, the following equation is obtained

$$\frac{f_{c_x}\delta}{k} = 0.765\left(\frac{3600c_p\mu g}{k}\right) = 0.765Pr$$

or

$$f_{c_x} = \frac{0.765}{\delta} kPr = \frac{0.765k}{\delta} \tag{21}$$

because the Prandtl modulus was assumed to be unity in order to obtain equation (11).

Equation (21) suggests that heat is conducted through the boundary layer by conduction across the thickness δ. Physically this is not the case; heat is transferred into the boundary layer and carried within the boundary layer along the surface of the plate. No heat is transferred to the free air stream by transfer across the boundary layer.

In order to account approximately for the effect of the pressure gradient existing about the airfoil, the thickness of the laminar boundary layer δ is computed by substituting the Blasius distribution in the von Kármán momentum condition and integrating. This operation yields an equation for δ^2 of the form (reference 10)

$$\delta^2 = \frac{5.3c^2}{Re_c}\left(\frac{x/c}{U/u_\infty}\right)\frac{\int_0^{x/c}\left(\frac{U}{u_\infty}\right)^{8.17} d\left(\frac{x}{c}\right)}{\left(\frac{U}{u_\infty}\right)^{8.17}\left(\frac{x}{c}\right)} \tag{22}$$

Equation (22) for the boundary-layer thickness represents the point in the derivation of Allen and Look where the treatment differs from the case of heat transfer in a laminar boundary layer along a flat plate.

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[3]The thickness of the laminar boundary layer δ is arbitrarily defined as the distance from the solid-fluid interface to a point in the boundary layer where the dynamic pressure is one-half of its value outside the boundary layer. (See appendix B.)

Although the Blasius velocity profile which is only applicable to conditions of flow along a flat plate $\left(\text{that is, } \frac{\partial p}{\partial x} = 0\right)$ has been used, substitution of the equation for the Blasius velocity profile into the momentum equation containing a term involving the pressure gradient should yield a value of δ more nearly representative of conditions at the surface of an airfoil.

Combination of equations (21) and (22) yields the final equation:

$$\frac{f_{c_x}}{3600 c_p \gamma U} = \frac{0.332}{\sqrt{Re_x}} \left[\frac{\int_0^{x/c} \left(\frac{U}{u_\infty}\right)^{8.17} d\left(\frac{x}{c}\right)}{\left(\frac{U}{u_\infty}\right)^{8.17} \left(\frac{x}{c}\right)} \right]^{-1/2} \tag{23}$$

The equation is written in this manner to allow ready comparison with flat-plate heat-transfer equations. The Prandtl modulus does not appear in equation (23) because it was initially postulated to be unity.

<u>Method of Frick and McCullough</u>.– The method of Frick and McCullough (reference 11) for treating the heat transfer through laminar boundary layers is similar to that of Allen and Look but is generalized to include values of Pr other than unity by utilizing Pohlhausen's exact solution of the differential equations for heat transfer in a laminar boundary layer along a flat plate.

As discussed in the introduction, the Pohlhausen solution yields the expression:

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \left(\frac{\partial u}{\partial y}\right)_{y=0} \sqrt[3]{Pr}\, \frac{T_o - T_\infty}{U}$$

As in the method of Allen and Look the velocity gradient at the wall is derived from the Blasius solution for a flat plate for the condition that at $y = \delta$, $\frac{u}{U} = 0.707$. (See footnote 3.)

Thus

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{0.765}{\delta} U \tag{24}$$

therefore

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0.765 \sqrt[3]{Pr}\, \frac{T_o - T_\infty}{\delta} \tag{25}$$

Thus

$$\frac{q_x}{A} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = f_{c_x}\left(T_o - T_\infty\right) = 0.765\ \sqrt[3]{Pr}\ \frac{k}{\delta}(T_o - T_\infty) \tag{26}$$

or, for a magnitude[4] of $Pr = 0.760$

$$f_{c_x} = 0.700\ \frac{k}{\delta} \tag{27}$$

where δ, as in the Allen and Look report, is given by equation (22). The substitution of this value of δ then partly accounts for the flow conditions along surfaces other than flat ones.

The difference between the coefficient 0.700 as found by Frick and McCullough and the value of 0.765 found by Allen and Look arises from the presence of the cube root of Pr in equation (26).

In general, for other magnitudes of Pr, by substitution of equation (22) into equation (26)

$$\frac{f_{c_x}}{3600c_p\gamma U} = \frac{0.332}{\sqrt{Re_x}}\ (Pr)^{-2/3}\left[\frac{\int_0^{x/c}\left(\frac{U}{u_\infty}\right)^{8.17} d\left(\frac{x}{c}\right)}{\left(\frac{U}{u_\infty}\right)^{8.17}\left(\frac{x}{c}\right)}\right]^{-1/2} \tag{28}$$

Basically this solution is identical to that of Allen and Look, with the exception of the use of the Pohlhausen solution instead of the Reynolds solution for heat transfer from a heated plate.

<u>Method of Martinelli, Boelter, and others</u>.- Probably the simplest approximate method of computing the point unit thermal conductance of the laminar boundary layer over wings is that of Martinelli, Guibert, Morrin, and Boelter (reference 13). The airfoil surface is conceived as a combination of a cylinder close to the leading edge and a flat plate beyond; the known data and equations for the heat transfer from these surfaces are then applied to the ideal surfaces.

(a) Near stagnation point

For the magnitude of the unit thermal conductance at the stagnation point of a cylinder, Squire's analytical solution of the

[4] A summary of data (reference 12) reveals that the value of Pr for the temperatures usually encountered in wing anti-icing is closer to 0.72. A magnitude of $Pr = 0.72$ is thus utilized in all laminar airfoil calculations in the present report.

differential equations for the boundary layer (reference 7, vol. II, p. 631) yields

$$Nu_{stag} = 1.14Pr^{0.4}Re_D^{0.5} \tag{29}$$

The point conductance along the leading edge at any angle (up to about $\varphi = 90^{\circ}$) measured from the stagnation point (reference 14) can then be approximated by the equation

$$\frac{f_{c_\varphi}D}{k} = 1.14Pr^{0.4}Re_D^{0.5}\left[1-\left(\frac{\varphi}{90}\right)^3\right] \tag{30}$$

or if the properties of air are expressed by a power function in the absolute temperature T

$$f_{c_\varphi} = 0.194T^{0.49}\left(\frac{u_\infty\gamma}{d}\right)\left[1-\left(\frac{\varphi}{90}\right)^3\right] \tag{31}$$

(b) Remainder of wing (laminar flow)

For the heat transfer along the portion of the airfoil section which is considered to behave as a flat plate, the equation of Colburn based upon Pohlhausen's analytical solution is used, (reference 15)

$$\frac{f_{c_x}}{3600C_p\gamma U}(Pr)^{2/3} = 0.332Re_x^{-0.5} \tag{19}$$

where U is the velocity near the airfoil surface at the point x, computed from the pressure distribution existing about the airfoil. (See appendix B.) The coordinate x is measured from the stagnation point. When the properties of air are expressed as a function of temperature

$$f_{c_x} = 0.0562T^{0.50}\left(\frac{U\gamma}{x}\right)^{0.50} \tag{32}$$

For comparison with the previous two methods, equation (19) may be written as

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{0.332}{\sqrt{Re_x}}(Pr)^{-2/3} \tag{33}$$

The heat-transfer equation (30) used in Martinelli's method for calculating the point conductance near the forward stagnation point is obtained from analysis and data on heat transfer near the stagnation point of a cylinder. The equations used beyond the leading edge in the method are, of course, purely flat-plate relations, but the substitution of the actual velocity U at the edge of the boundary layer for u_∞ is intended to account approximately for the effect of the pressure distribution along an airfoil surface.

Method of Squire.- The procedure adopted by Squire (reference 16) for the calculation of the point unit thermal conductance over the outer surface of an airfoil involves more lengthy computation than any of the preceding methods, but probably represents the most rational analysis of the aerodynamic and thermal relations thus far published.

It can be shown that when the Peclet modulus $(u_\infty d/a)$ is large, the conditions with respect to the temperature distribution near the surface of a heated body past which fluid is flowing are the same as those with respect to the velocity distribution when the Reynolds modulus is large. That is, a "thermal boundary layer" of small thickness exists near the surface, in which the temperature falls rapidly from its value at the surface to the temperature in the free stream. Thus it follows to define a "thermal displacement thickness" δ_2 for the thermal boundary layer in a manner analogous to the hydrodynamic case, (appendix B) by the equation,

$$\delta_2 = \int_0^\infty \frac{T - T_\infty}{T_0 - T_\infty}\, dy \tag{34}$$

in which T, T_0, and T_∞ are the temperatures in the boundary layer, at the surface, and in the free stream, respectively.

In Squire's derivation an hypothesis of fundamental importance is made; namely that the temperature distribution is of a similar form to that of velocity. It should be clearly borne in mind that when this postulate is made it is assumed that the pressure gradient affects the temperature distribution in exactly the same manner as it does the velocity distribution. This is not strictly correct, and the accuracy of the approximation is still open to experimental verification. (See appendix C.)

With this hypothesis, utilizing the Blasius series solution for the velocity distribution in the laminar boundary layer of a flat plate, there is obtained

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{0.5715}{\delta_2}\,(T_0 - T_\infty) \tag{35}$$

or

$$f_{c_x} = \frac{0.5715}{\delta_1}k\left(\frac{\delta_1}{\delta_2}\right) \tag{36}$$

The equation for the hydrodynamic-boundary-layer displacement thickness (appendix B) may be written as (reference 17)

$$\delta_1{}^2 = \frac{2.960}{U}\nu x \frac{\int_0^x U^5\,dx}{xU^5} \tag{37}$$

so that

$$f_{c_x} = \frac{0.5715}{\sqrt{2.960}}\,k\left(\frac{\delta_1}{\delta_2}\right)\sqrt{\frac{Ux}{\nu}}\left(\frac{\int_0^x U^5\,dx}{xU^5}\right)^{-1/2} \tag{38}$$

or

$$\frac{f_{c_x}}{3600c_p\gamma U} = \frac{0.332}{\sqrt{Re_x}}\left(\frac{\delta_1}{\delta_2}\,\frac{1}{Pr}\right)\left(\frac{\int_0^x U^5\,dx}{xU^5}\right)^{-1/2} \tag{39}$$

The ratio $\frac{\delta_1}{\delta_2}$ appearing in equation (39) is a function of U and Pr and was obtained by Squire by means of the integral heat balance derived in section entitled "Heat balance." When the Blasius distribution for velocity and temperature is substituted in this heat balance δ_2 may be expressed in terms of the known value of δ_1 by the relation

$$\left(\frac{\delta_2}{\delta_1}\right)^2 \varphi\left(\frac{\delta_2}{\delta_1}\right) = \frac{0.3681}{Pr}\,\frac{U^4\int_0^x U\,dx}{\int U^5\,dx} \tag{40}$$

where φ is a known function of its argument which Squire tabulates for values of $\frac{\delta_2}{\delta_1}$ between 0.5 and 2.0. If the pressure gradients are not too large, the ratio $\frac{\delta_2}{\delta_1}$ very nearly equals $Pr^{1/3}$.

It is important that one recognize wherein Squire's method is similar to the approach used by Look and Allen, Frick and McCullough, and Martinelli and in what way it differs from these. It will be recalled that in the latter three methods the authors use the hydrodynamic equations and heat-transfer equations for a flat plate and include the effect of the pressure gradient only in computing the value of the boundary-layer thickness, which appears in their final equations for the local unit thermal conductance, and by substituting the actual velocity at the edge of the boundary layer. Although it is true that Squire also used the flow and heat-transfer equations which are valid only for a flat plate, his method represents an advance in that it uses a corrected value of the thermal-boundary-layer thickness computed as a function of the hydrodynamic-boundary-layer thickness in the final equation for f_{c_x}.

Method of this report.- It will be noted that each of the preceding methods involves a compromise or combination of hydrodynamic and heat-transfer equations which are strictly applicable only in the case of boundary layers along plane surfaces, together with the momentum equation which is valid for any curved surface up to the point of separation, to yield final expressions for the unit thermal conductance over an airfoil shape.

The authors of the present report have attempted an analysis of the problem based on relations true for any boundary layer regardless of the shape of the section and without recourse to the Blasius flat-plate solution.

Two such general expressions are available: one being merely an integrated heat balance across the boundary layer, and the other an integrated force balance. If one postulates steady flow, an incompressible fluid, and no dissipation of kinetic energy, the first, often called the "energy equation" of the boundary layer, is

$$\frac{\partial}{\partial x}\int_0^{\delta_H} uT\,dy - T_1\,\frac{\partial}{\partial x}\int_0^{\delta_H} u\,dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (41)$$

and the second is the well-known von Kármán momentum equation (appendix B)

$$\frac{\partial}{\partial x}\int_0^{\delta} u^2\,dy - U\,\frac{\partial}{\partial x}\int_0^{\delta} u\,dy = -\frac{\delta}{\rho}\,\frac{\partial p}{\partial x} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (42)$$

When a fourth power polynomial is postulated to represent both the velocity and temperature distribution in the boundary layer, the following equation is obtained with the use of the proper boundary conditions. (See appendix A.)

$$\frac{u}{U} = \left(2 + \frac{1}{6}\lambda\right)\left(\frac{y}{\delta}\right) + \left(-2 + \frac{1}{2}\lambda\right)\left(\frac{y}{\delta}\right)^3 + \left(1 - \frac{1}{6}\lambda\right)\left(\frac{y}{\delta}\right)^4 \tag{43}$$

where

$$\lambda = \frac{\delta^2}{\nu}\frac{dU}{dx}$$

$$\frac{T}{T_1} = 2\left(\frac{y}{\delta_H}\right) - 2\left(\frac{y}{\delta_H}\right)^3 + \left(\frac{y}{\delta_H}\right)^4 \tag{44}$$

Equation (41) may be written in the following form:

$$\frac{\partial}{\partial x} UT_1 \int_0^{\delta_H} \frac{u}{U}\frac{T}{T_1}\,dy - T_1 \frac{\partial}{\partial x} U \int_0^{\delta_H} \frac{u}{U}\,dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{45}$$

which, upon substitution of the polynomials for the velocity and temperature distribution and integrating, yields

$$\frac{\partial}{\partial x}(U\delta_H\varphi) = -\frac{a}{T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{46}$$

where

$$\varphi = \left[(-0.134 - 0.011\lambda)Pr^{-1/3} + (0.021 - 0.006\lambda)Pr^{-1} + (-0.006 + 0.001\lambda)Pr^{-4/3}\right] \tag{47}$$

and

$$\frac{\delta_H}{\delta} \approx Pr^{-1/3} \quad \text{(See appendix A.)} \tag{48}$$

Therefore

$$f_{c_x} = 3600\gamma C_p \frac{\partial}{\partial x}(U\delta_H\varphi) \tag{49}$$

Details of the method of derivation and the procedure for calculating the point conductance in this method will be found in appendix A.

It was previously pointed out that Squire's method represented a closer correspondence to physical facts than those previously proposed in that a general energy balance on the boundary layer was used to calculate the ratio of the thermal- to the hydrodynamic-boundary-layer thickness. The method does, however, employ the Blasius series solution for the flat plate to represent the velocity distribution in the boundary layer. In the aforementioned method this latter approximation is eliminated by using both fluid-flow and heat-transfer equations which are general for the laminar boundary layer along any type of section and which take into account the pressure distribution actually existing over the airfoil surface. Specifically, the Pohlhausen polynomial is used in preference to the Blasius series to represent the velocity distribution in the boundary layer.

In so doing, the limitations of the Pohlhausen method should be clearly borne in mind (appendix B). It is generally conceded that this method gives a good representation of the velocity distribution in a laminar boundary layer in which the fluid is being accelerated and is fairly accurate in regions of retarded flow at positions distant from the point of separation. The method becomes less and less accurate as the point of separation is approached and breaks down completely upon reaching that point.

Summary of methods.- The final equations for the laminar point unit thermal conductance for each of the four methods previously published are as follows:

Allen and Look:

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{0.332}{\sqrt{Re_x}}\left[\frac{\int_0^{x/c}\left(\frac{U}{u_\infty}\right)^{8.17} d\left(\frac{x}{c}\right)}{\left(\frac{U}{u_\infty}\right)^{8.17}\left(\frac{x}{c}\right)}\right]^{-1/2} \tag{23}$$

Frick and McCullough:

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{0.332}{\sqrt{Re_x}} Pr^{-2/3}\left[\frac{\int_0^{x/c}\left(\frac{U}{u_\infty}\right)^{8.17} d\left(\frac{x}{c}\right)}{\left(\frac{U}{u_\infty}\right)^{8.17}\left(\frac{x}{c}\right)}\right]^{-1/2} \tag{28}$$

Martinelli and others:

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{0.332}{\sqrt{Re_x}} Pr^{-2/3} \tag{33}$$

Squire:

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{0.332}{\sqrt{Re_x}}\left(\frac{\delta_1}{\delta_2}\frac{1}{Pr}\right)\left(\frac{\int_0^x U^5\, dx}{xU^5}\right)^{-1/2} \tag{39}$$

Comparison of equations (23), (28), (33), and (39) reveals that:

(1) Equations (28), (33), and (39) account for variations of Pr from unity, but equation (23) does not.

(2) All four equations account approximately for the pressure gradient existing along the airfoil by substituting the velocity at the edge of laminar boundary layer for u_∞ in flat-plate relations but, in addition, equations (23), (28), and (39) make further approximate corrections by means of the terms in brackets.

(3) Equations (28) and (33) are identical for the case of heat transfer over plane surfaces, and all the equations are identical for heat transfer from a flat plate at $Pr = 1$.

The next section of this report is devoted to a discussion of heat flow into turbulent boundary layers. A generalized discussion of theory is followed by details of four calculation procedures.

Turbulent Regime

Explorations of the velocity immediately adjacent to streamlined surfaces along which a turbulent boundary layer exists are not numerous, but measurements of the velocity near the walls of tubes in which a fluid is flowing turbulently have been accurately performed (references 18 to 20) and reveal that a completely turbulent boundary layer probably does not exist. Near the walls of the tube there is found to be a laminar "sublayer" in which the flow remains viscous even though the fluid far from the surface flows turbulently. For purposes of analysis a transition layer, sometimes called the "buffer layer" may be visualized as existing between this laminar sublayer and the turbulent fluid. In the laminar sublayer viscous forces predominate; in the turbulent region "eddy" forces are controlling; whereas in the buffer layer both viscous and eddy forces are of the same order of magnitude. Reynolds (reference 21) postulated that in the turbulent region, momentum and heat are transferred by similar mechanisms. In the laminar sublayer, molecular heat transfer occurs, whereas in the buffer layer both molecular and "eddy" heat transfer take place.

On the basis of these concepts, a boundary-layer heat-transfer equation and a boundary-layer hydrodynamic equation have been written (reference 7, vol. II, p. 650). The heat-transfer equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\nu}{Pr}\frac{\partial T}{\partial y} + \epsilon_H\frac{\partial T}{\partial y}\right) \tag{50}$$

where ϵ_H is the so-called "eddy diffusivity" for heat.

The hydrodynamic equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y} + \epsilon_M\frac{\partial u}{\partial y}\right) \tag{51}$$

where ϵ_M is the so-called eddy diffusivity for momentum. The velocities u and v in equations (50) and (51) are mean values with respect to time.

On the basis of the momentum-transfer theory exact similarity between the temperature and velocity distributions therefore exists if

(1) The eddy diffusivity for heat ϵ_H equals the eddy diffusivity for momentum ϵ_M. (This statement is the original basis for the Reynolds analogy as developed by Stanton (reference 22).)

(2) The Prandtl modulus equals unity. (This statement is necessary because of the existence of the laminar sublayer and buffer layer, in which viscous stresses and molecular heat transfer are important.)

(3) The pressure gradient $\frac{\partial p}{\partial x}$ is zero.

If all these conditions are satisfied and if the temperature and velocity distributions have the same boundary conditions, the solutions for velocity and temperature will be identical. Thus, for turbulent flow, as in the case of laminar flow:

$$\frac{\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\left(\frac{\partial T}{\partial y}\right)_{y=0}} = \frac{u_\infty}{T_0 - T_\infty} \tag{52}$$

As in the case of a laminar boundary layer, this equation can be rewritten as:

$$\frac{f_{c_x}}{3600 C_p \gamma u_\infty} (Pr) = \frac{C_{f_x}}{2} \tag{53}$$

or since $Pr = 1$

$$\frac{f_{c_x}}{3600 C_p \gamma u_\infty} = \frac{C_{f_x}}{2} \tag{54}$$

This equation is usually referred to as Reynolds analogy and, as written, applies only to heat transfer from plates $\left(\frac{\partial p}{\partial x} = 0\right)$ at a uniform temperature to a fluid with $Pr = 1$. Modifications of the analogy to apply for other magnitudes of Pr and for flow conditions in which a pressure gradient exists will be discussed below.

Modification of Reynolds analogy to account for pressure gradient $\frac{\partial p}{\partial x}$.— If the pressure gradient $\frac{\partial p}{\partial x}$ is not zero, then equations (50) and (51) are no longer analogous and the exact equivalence of u and T, even for $Pr = 1$, breaks down.

Calculations for an airfoil on the basis of a heat balance in which the temperature and velocity distributions in the turbulent boundary layer are assumed to vary identically with pressure gradient (reference 16) reveal that the pressure gradient may have a large effect on the rate of heat transfer. The calculations presented in reference 16, however, probably overemphasize the effect of $\frac{\partial p}{\partial x}$ because actually, the pressure gradient influences the velocity much more than it does the temperature, as noted by the fact that the term $\frac{\partial p}{\partial x}$ enters the velocity equation directly, but influences the temperature only through its effect on the velocities u and v.

Thus the exact influence of $\frac{\partial p}{\partial x}$ on the rate of heat transfer is difficult to establish; one may probably say that the effect should not be pronounced, but rather of secondary importance. For lack of a simultaneous solution of the heat-transfer and hydrodynamic equations for the turbulent boundary layer, including the effect of pressure gradient, approximate corrections to the Reynolds analogy (similar to those made for the laminar boundary layer) to account for $\frac{\partial p}{\partial x}$ have been presented in the literature. In general, these approximate corrections consist of substituting U, the velocity near the airfoil surface (calculated by means of the pressure distribution existing about the airfoil; see appendix B), for u_∞ in equation (54) and calculating the drag coefficient C_{f_x} for the airfoil at the point in question. The Reynolds analogy then becomes:

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{C_{f_x}}{2} \tag{55}$$

<u>Modification of Reynolds analogy for magnitudes of Pr other than unity</u>.– Since the Prandtl modulus for air is less than unity, the Reynolds analogy must be modified to allow calculations for other magnitudes of the Prandtl modulus. Several methods are available, all being based on analyses of heat transfer to fluids in tubes:

(a) Von Kármán modification: By analyzing the heat transfer from a tube to an enclosed fluid flowing turbulently, von Kármán obtained (reference 23):

$$\frac{f_c}{3600C_p\gamma u_m} = \frac{\frac{C_f}{2}}{1 + 5\sqrt{\frac{C_f}{2}}\left\{(Pr - 1) + \log_e\left[1 + \frac{5}{6}(Pr - 1)\right]\right\}} \tag{56}$$

(b) Boelter, Martinelli, and Jonassen modification: By extending the von Kármán analysis to include a more precise consideration of the turbulent region, Boelter and others obtained for flow in tubes only (reference 24):

$$\frac{f_c}{3600C_p\gamma u_m} = \frac{\sqrt{\frac{C_f}{2}}\,\frac{\Delta T_{max}}{\Delta T_{mean}}}{5\left[Pr + \log_e(1 + 5Pr) + 0.5\log_e\frac{Re}{60}\sqrt{\frac{C_f}{2}}\right]} \tag{57}$$

(c) Colburn, by empirical correlation of data on heat transfer from flat plates, obtained (reference 15):

$$\frac{f_c}{3600C_p\gamma u_m}Pr^{2/3} = \frac{C_f}{2} \tag{58}$$

Numerical calculation of the von Kármán and the Boelter method reveals that for a range of Pr between 0.5 and 10, the complex expressions involving Pr reduce to that of Colburn with fair accuracy (reference 24). Thus, for fluids with $0.5 < Pr < 10$ the modified Reynolds analogy may be rewritten as:

$$\frac{f_{c_x}}{3600 C_p \gamma U} Pr^{2/3} = \frac{C_{f_x}}{2} \tag{59}$$

When the equation for the local drag coefficient along a flat plate (reference 7, vol. II, p. 362)

$$\frac{C_{f_x}}{2} = 0.0296 Re_x^{-0.2} \tag{60}$$

is substituted into equation (59), the expression becomes:

$$\frac{f_{c_x}}{3600 C_p \gamma U} = 0.0296 Pr^{-2/3} Re_x^{-0.2} \tag{61}$$

This equation is called "Colburn's" turbulent heat-transfer equation in the remainder of this report.

Heat balance.- If the velocity and temperature distributions are accurately known in the boundary layer, a heat balance will yield a simple method of obtaining the unit thermal conductance from the airfoil. As in the case of laminar flow

$$\frac{f_{c_x}}{3600 C_p \gamma U} = \frac{1}{U} \frac{\partial}{\partial x} \left[U \int_0^\infty \frac{u}{U} \frac{(T - T_\infty)}{(T_o - T_\infty)} dy \right] \tag{19a}$$

The difficulty in application of this method lies in the necessity for accurate knowledge of the velocity and temperature distributions. The methods based on the Reynolds analogy and its modifications require only a knowledge of the velocity gradient at the surface of the solid, as may be seen from an inspection of equation (52).

Method of Frick and McCullough.- Utilizing the Reynolds analogy,

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \left(\frac{\partial u}{\partial y}\right)_{y=0} \left(\frac{T_o - T_\infty}{U}\right) \tag{62}$$

Frick and McCullough obtain

$$\frac{q_x}{A} = c_p \tau_o \left(\frac{T_o - T_\infty}{U}\right) \tag{63}$$

In order to apply this equation to heat transfer from airfoil surfaces, the local shear τ_o is calculated by means of the von Kármán expression for the skin friction over a flat plate with a fully developed turbulent boundary layer. Thus

$$\tau_o = \frac{\rho U^2}{\zeta^2} \tag{64}$$

where

$$\zeta = 2.557 \log_e \left(4.075 \frac{U\theta}{\nu}\right) \tag{65}$$

in which the momentum thickness θ is calculated from airfoil boundary-layer data. (See appendix B.) Substituting for τ_o in equation (63) and rearranging yields

$$\frac{q_x}{A} = \frac{k}{c\zeta^2} \mathrm{Pr}\mathrm{Re}_c\left(\frac{U}{u_\infty}\right)(T_\infty - T_o) \tag{66}$$

If one considers $\dfrac{\zeta^2 c}{\mathrm{Re}_c\left(\dfrac{U}{u_\infty}\right)} = \delta_T$ a characteristic length for the turbulent boundary layer, then

$$\frac{q_x}{A} = \frac{k}{\delta_T} \mathrm{Pr}\,(T_\infty - T_o) \tag{67}$$

or

$$f_{c_x} = \frac{k}{\delta_T} \mathrm{Pr}$$

Letting[5] $Pr = 0.760$, Frick and McCullough obtain[6]

$$f_{c_x} = \frac{0.760k}{\delta_T} \tag{68}$$

This characteristic length δ_T should not be confused with the thickness of the turbulent boundary layer. For a flat plate this latter quantity is given by the expression (reference 7, vol. II, p. 362)

$$\frac{\delta}{x} = 0.37 Re_x^{-0.2} \tag{69}$$

from which it is seen that δ_T is approximately proportional to x and inversely proportional to the boundary-layer thickness.

In summary it may be stated that the treatment of turbulent-boundary-layer heat transfer in the report of Frick and McCullough assumes complete equivalence of skin friction and heat transfer and uses the best method known for the calculation of turbulent skin friction.

In order to compare the available methods for calculating heat transfer into turbulent boundary layers, equation (68) is rewritten for ready comparison with the flat-plate equations to be derived in the next section. The equation of Frick and McCullough becomes:

$$\frac{f_{c_x}}{3600 c_p \gamma U} = 0.0296 Re_x^{-0.2} \left(\frac{Re_x^{0.2}}{0.0296 \zeta^2} \right) \tag{70}$$

The Prandtl modulus does not appear in equation (70) because a magnitude of $Pr = 1$ was tacitly assumed in its derivation.

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[5] As in the case of the laminar sublayer $Pr = 0.72$ is more correct and is utilized in the calculation of this report.

[6] It should be kept in mind that equation (68) is based on equation (62). Equation (62) does not include Pr, but in its derivation a Pr value of unity is postulated. In order to obtain equation (63) from equation (62) a Prandtl modulus of unity is utilized. In order to obtain equation (67) from equation (63), equation (63) was multiplied and divided by Pr Thus, as long as proper values of μ, C_p, and k are utilized in equation (67), substitution of the proper magnitude of Pr, even though differing from unity, will yield a value of f_{c_x} which in reality is calculated for $Pr = 1$ and is therefore too low. In a later section the equations of Frick and McCullough are modified to correct approximately for magnitudes of Pr not equal to unity.

Method of Martinelli and others.– For the case of turbulent flow beyond the leading edge, the Colburn equation for turbulent heat transfer from flat plates is utilized. Thus

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296Pr^{-2/3}Re_x^{-0.2} \tag{71}$$

or expressing the properties of air approximately by a power function in T the absolute temperature,

$$f_{c_x} = 0.51T^{0.3}\left(\frac{U\gamma}{x^{0.25}}\right)^{0.80} \tag{72}$$

When equations (70) and (71) are compared two points of difference are noted:

(1) The term in parentheses in equation (70) represents an approximate correction to the flat-plate equation for the pressure gradient existing about the airfoil, in addition to the use of the velocity near the airfoil surface in the flat-plate Reynolds analogy.

(2) Equation (70) does not involve Pr, whereas equation (71) includes $Pr^{-2/3}$. Since the latter equation accounts for the Prandtl modulus more accurately than the former, the equation of Frick and McCullough, to be more correct, should be written as:

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296Pr^{-2/3}Re_x^{-0.2}\left(\frac{Re_x^{0.2}}{0.0296\zeta^2}\right) \tag{73}$$

Magnitudes of f_{c_x} calculated from the original Frick and McCullough equation are thus too low by the factor $Pr^{-2/3}$. Equation (73) is called the "modified Frick and McCullough equation" in this report.

Methods of Squire.– Squire (reference 16) presents two methods of calculating f_{c_x} for an airfoil. The first method employs the Reynolds analogy as modified by von Kármán. Thus

$$\frac{f_{c_x}}{3600C_p\gamma U} = \frac{\frac{C_{f_x}}{2}}{1 + 5\sqrt{\frac{C_{f_x}}{2}}\left\{(Pr - 1) + \log_e\left[1 + \frac{5}{6}(Pr - 1)\right]\right\}} \tag{74}$$

where $\frac{C_{f_x}}{2}$ is calculated at each point along the airfoil by utilizing the von Kármán expression for skin friction along a flat plate with a fully developed turbulent boundary layer, as outlined in the method of Frick and McCullough.

In order to compare equation (74) with those for flat plates, it may be rewritten as:

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296{Re_x}^{-0.2}\left(\frac{1}{1 + \frac{5}{\zeta}\left\{(Pr - 1) + \log_e\left[1 + \frac{5}{6}(Pr - 1)\right]\right\}}\right)\left(\frac{{Re_x}^{0.2}}{0.0296\zeta^2}\right) \tag{75}$$

The first term in parentheses corrects the Reynolds analogy for the flat plate for magnitudes of Pr other than unity. Numerically (for small values of Pr) the term is approximately equal to $Pr^{-2/3}$. The last term in parentheses represents the approximate correction to the flat-plate equation for the pressure gradient existing about the airfoil and is identical to that utilized by Frick and McCullough.

To indicate the probable effect of pressure gradients, Squire presents an alternative form of Reynolds analogy by assuming that the temperature and velocity distributions are exactly similar in the presence of such gradients.

The energy equation for the boundary layer is

$$\frac{\partial}{\partial x}\left[\int_0^\infty u(T - T_\infty)\,dy\right] = \frac{q_x}{3600C_p\gamma} \tag{76}$$

When it is assumed that the temperature and velocity distributions are similar ($Pr = 1$)

$$\frac{T - T_\infty}{T_o - T_\infty} = 1 - \frac{u}{U} \tag{77}$$

equation (76) becomes

$$\frac{f_{c_x}}{3600c_p\gamma U} = \frac{1}{U}\frac{\partial}{\partial x}\left[U\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right] \tag{78}$$

$$= \frac{d\theta}{dx} + \frac{U'}{U}\theta$$

where θ is the momentum thickness of the boundary layer and equals $\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$, (see appendix B) and the primes denote differentiation with respect to x. From the momentum theorem, however,

$$\frac{d\theta}{dx} = \frac{\tau_o}{\rho U^2} - \frac{U'}{U}\theta\left(\frac{\delta}{\theta} + 2\right) \tag{79}$$

so that

$$\frac{f_{c_x}}{3600c_p\gamma U} = \frac{\tau_o}{\rho U^2} - \frac{U'}{U}\theta\left(\frac{\delta}{\theta} + 1\right) \tag{80}$$

As has been mentioned previously, a plot of f_{c_x} against x/c for the usual Reynolds analogy and for this latter modification indicating the effect of $\frac{\partial p}{\partial x}$ shows large deviations for an airfoil example at $\frac{x}{c} > 0.4$.

Because $\frac{\tau_o}{\rho U^2} = \frac{C_{f_x}}{2}$, equation (80) then directly indicates the comparison between the additional correction for pressure gradient and the usual Reynolds analogy for a flat plate.

Also since the momentum theorem (equation (79)) and the energy balance are equally valid for laminar and turbulent boundary layers, equation (80) applies as well to the laminar case, and the term to the right of the minus sign affords an indication of the relative effect of pressure gradient on the heat transfer into laminar and turbulent boundary layers.

In order to compare equation (80) with the turbulent equation of Frick and McCullough and the flat-plate relations, it may be written in the form

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296{Re_x}^{-0.2}\left(\frac{{Re_x}^{0.2}}{0.0296\zeta^2}\right) - \left[\frac{U'}{U}\left(\frac{\delta}{\theta} + 1\right)\theta\right] \tag{81}$$

Summary of methods.- The final equations for the point unit thermal conductance for turbulent boundary layers, are as follows:

Frick and McCullough (method I)

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296{Re_x}^{-0.2}\left(\frac{{Re_x}^{0.2}}{0.0296\zeta^2}\right) \tag{70}$$

Modified Frick and McCullough (method II)

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296{Re_x}^{-0.2}Pr^{-2/3}\left(\frac{{Re_x}^{0.2}}{0.0296\zeta^2}\right) \tag{73}$$

Martinelli and others:

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296{Re_x}^{-0.2}Pr^{-2/3} \tag{71}$$

Squire (method I)

$$\frac{f_{c_x}}{3600C_p\gamma U} = 0.0296{Re_x}^{-0.2}\left(\frac{1}{1 + \frac{5}{\zeta}\left\{(Pr - 1) + \log_e\left[1 + \frac{5}{6}(Pr - 1)\right]\right\}}\right)\left(\frac{{Re_x}^{0.2}}{0.0296\zeta^2}\right) \tag{75}$$

Squire (method II)

$$\frac{f_{c_x}}{3600c_p\gamma U} = 0.0296Re_x^{-0.2}\left(\frac{Re_x^{0.2}}{0.0296\zeta^2}\right) - \left[\frac{U'}{U}\left(\frac{\delta}{\theta} + 1\right)\theta\right] \qquad (81)$$

The bracketed terms in these equations directly indicate the corrections made for pressure gradient and variations in the Prandtl modulus.

Comparison of equations (70), (73), (71), (75), and (81) reveals that:

(1) Equations (73), (71), and (75) account for variations of Pr from unity, but equations (70) and (81) do not.

(2) All five equations account approximately for the pressure gradient existing along the airfoil by substituting the velocity near the airfoil surface U for u_∞ in the flat-plate relations. In addition, equations (70), (73), and (75) make further approximate corrections for the variation of point drag coefficient along the airfoil surface. Finally, equation (81) includes a further corrective term which results from the calculation of a heat balance on the boundary layer. The last equation probably overemphasizes the role of the pressure gradient.

(3) Equations (70) and (71) are identical for heat transfer from a flat plate and have been checked experimentally for this case. Equation (75) is practically identical to equations (71) and (73) for heat transfer over a flat plate, and basically probably accounts for variations of Pr from unity more precisely than equations (71) and (73) which are based on an empirical correlation of experimental data.

(4) Although equations (70) and (71) are identical when $\frac{\partial p}{\partial x} = 0$, the results from these equations are strictly applicable only to fluids with Pr = 1.

(5) All equations are identical for heat transfer from a flat plate to a fluid with Pr = 1.

DISCUSSION OF NUMERICAL EXAMPLES

Methods Employed in Laminar Regime

In order to compare the various methods described for calculating the laminar point unit thermal conductance over wings, an airfoil shape was selected whose aerodynamic characteristics are very accurately known.

Such a section, for which the pressure and velocity distribution has been calculated theoretically and thoroughly tested experimentally, is a Joukowski profile whose characteristics are shown in figure 2. This particular profile is called "aerofoil A" by Bairstow and numerous tables of data concerning it will be found in reference 25.

For purposes of calculation, an airfoil with a chord of 7.78 feet was chosen which was to be maintained at a constant temperature of 70° F, while moving with an angle of attack of 1.5° at a velocity of 253 feet per second through air at a uniform temperature of 30° F. The values of f_{c_x} computed by the different methods for this example are shown graphically in figure 3.

It is seen that all of the methods discussed give answers that are in fairly good agreement. The reasons for differences are readily observed by inspection of table I in which the various terms of equations (23), (28), (33), and (39) are presented. For example Allen and Look's values are obviously too low because the calculations were based on a Pr value of 1. The values calculated by equations (33) and (39) are in very good agreement, but some of this agreement is fortuitous, as table I reveals. Thus, the individual corrections of Squire for $\frac{\partial p}{\partial x}$ and Pr are of such a magnitude that their combined effect yields results which are in very close agreement with those of Martinelli and others; calculations on a different airfoil, however, may give results which are considerably more at variance.

If, as is often the case in practice, an approximate value of the external conductance, or a value indicating an order of magnitude, is desired, then the method of Martinelli is the simplest and most rapid and is usually of sufficient accuracy; in other cases the accuracy of the answer desired would determine the choice of method. It should again be remarked in this connection, however, that the example cited for purposes of comparison does not represent an extreme case of pressure gradient around an airfoil and that the deviations among the methods due to the effect of the pressure gradient on the heat transfer might be considerably greater for other types of shapes, such as thick wings, fuselages, and so forth.

Methods Employed in the Turbulent Regime

In order to compare the various methods for calculating the unity conductance along an airfoil for the turbulent boundary layer, the wing profile utilized by Frick and McCullough (NACA 65,2-016) was selected for calculation. The airfoil has a chord of 7 feet and was assumed to be moving with a velocity of 206 feet per second. The average temperature of the air in the turbulent boundary layer was taken as 40° F.

The five methods of calculation previously discussed were applied to the airfoil, assuming a turbulent boundary layer to exist from $x = 0$. The results are plotted in figure 4. It is noted that the five methods yield results, which, as in the case of the laminar-boundary-layer calculations, are in fair agreement. Certain differences are apparent, however, which merit further discussion.

In order to facilitate comparison of the various methods, the pertinent terms of equations (70), (73), (71), (75), and (81) are presented in table II. It is apparent from this table that Frick and McCullough's original equation is too low because of the tacit supposition that $Pr = 1$. Inclusion of the approximate correction for Pr in Frick and McCullough's methods raises the curve about 20 percent. The remaining methods check closely up to $\frac{x}{c} = 0.4$. At this point Squire's heat-balance method diverges rapidly from the other curves. Experimental evidence is not available to check this phenomenon, but it is probable that the heat-balance method proposed by Squire overemphasizes the importance of $\frac{\partial p}{\partial x}$.

The rather close comparison of the method proposed by Martinelli and others with the more refined techniques of the other authors is partly fortuitous, since for airfoils with abrupt pressure gradients the results from the various methods may be considerably more at variance.

Unit Conductance at the Stagnation Point

Allen and Look and also Frick and McCullough suggest the use of the approximation

$$\delta_{stag}{}^2 = \frac{c^2}{5Re_c}\frac{r}{c} \tag{82}$$

for computing the value of the point conductance at the stagnation point, where r is the radius of curvature of the leading edge. This equation considers the airfoil leading edge to be elliptical in form. The equation of the point conductance at the stagnation point then becomes

$$Nu_{stag} = \frac{f_c d}{k} = 2.42\sqrt{\frac{u_\infty d}{\nu}} = 2.42\sqrt{Re_D} \tag{83}$$

where d is twice the radius of curvature.

At the stagnation point the equations of Squire reduce to the form

$$\delta_1{}^2 = \frac{0.123d^2}{Re_D} \tag{84}$$

Since

$$f_{c_x} = \frac{0.5715}{\delta_1} k\left(\frac{\delta_1}{\delta_2}\right) \tag{36}$$

$$Nu_{stag} = \frac{f_c d}{k} = 1.63 \sqrt{Re_D}\left(\frac{\delta_1}{\delta_2}\right) \tag{85}$$

where $\frac{\delta_1}{\delta_2}$ is found by means of the equation

$$\left(\frac{\delta_2}{\delta_1}\right)^2 \varphi\left(\frac{\delta_2}{\delta_1}\right) = \frac{1.158}{Pr} \tag{86}$$

The term $\frac{\delta_1}{\delta_2}$ accounts both for variations in Pr from unity and for the pressure gradient about the stagnation point. At $Pr = 0.720$ equation (86) gives $\frac{\delta_1}{\delta_2} = 0.581$ so that

$$Nu_{stag} = 0.95 \sqrt{Re_D} \tag{87}$$

The equation proposed by Martinelli for the stagnation-point conductance is that derived in the theoretical analysis of heat transfer from the forward end of a cylinder, namely

$$Nu_{stag} = 1.14 Pr^{0.4} Re_D{}^{0.5} \tag{29}$$

or at $Pr = 0.720$

$$Nu_{stag} = 1.0 \sqrt{Re_D} \tag{88}$$

Comparison of equations (83), (87), and (88) immediately indicates the degree of correspondence between each of the three methods and the stagnation-point conductance for the leading edge of a cylinder. In the method of Martinelli, the flow at the leading edge of the airfoil is initially postulated to be exactly that of the flow at the forward portion of a cylinder and hence the airfoil stagnation-point conductance corresponds exactly with that of the cylinder. Apart from this method it is seen that Squire's method gives very much better agreement with data obtained at the stagnation point of a cylinder (reference 7, ch. IV) than does the method of Allen and Look.

If the cylinder-stagnation-point value of f_{c_x} is considered as being very close to the actual leading-edge value for the airfoil, then the error made by Allen and Look in setting the ratio $\frac{\delta_1}{\delta_2}$ equal to unity is well illustrated by the wide discrepancy between their value of the point conductance at the stagnation point with that obtained from analysis of heat transfer at the stagnation point where the pressure gradient is extreme. This discrepancy resulting from use of the Allen and Look method is an indication of the superiority of Squire's method when applied over the entire airfoil.

If in Allen and Look's method the leading edge of the airfoil is considered to be cylindrical instead of elliptical, their equations reduce to

$$\delta_{stag}{}^2 = \frac{0.289rc}{Re_c} \tag{89}$$

and

$$NU_{stag} = \frac{f_c d}{k} = 2.0\sqrt{Re_D} \tag{90}$$

rather than equations (82) and (83).

When the expressions for the point conductance at the stagnation point are applied to the Joukowski profile, which was used as an example in comparing the methods for the laminar regime, there is obtained:

Method	$f_{c_{x_{stag}}}$ (Btu/(hr)(sq ft)(°F))
Martinelli	109
Allen and Look (elliptical)	264
Allen and Look (cylindrical)	218
Squire	104

Propeller Calculation

The problem of heating propellers to prevent the formation of ice is becoming of increasing importance in the aircraft industry, and some method of calculating the point unit thermal conductance on the outer surface of the propeller would therefore be desirable.

The flow conditions around a propeller are complex, and the theory is rather imperfectly developed so that an exact analysis of the problem in a manner similar to that employed for an airfoil section in this report does not at present seem practicable.

It is well known, however, that a propeller may be considered as being made up of a series of airfoil elements, and a rough approximation to the variation of the point conductance radially and chordwise might therefore be obtained by calculating the chordwise distribution by any of the methods described herein for an airfoil shape, for each of these elements. Such a procedure was adopted here by applying Martinelli's equation for the laminar and turbulent cases to four different sections of a propeller shape whose characteristics are described by Bairstow (reference 25, p. 664).

Because experimental data on the pressure distribution around the blade elements of the propeller were lacking, the approximate equation of Seibert (reference 26)

$$U = u_\infty \left(1 \pm \frac{C_L}{4 \cos \alpha}\right) \tag{91}$$

+ for "upper" surface of airfoil

− for "lower" surface of airfoil

was used to calculate the velocity at the edge of the boundary layer.

The propeller in question was assumed to have a radius of 5 feet, to be rotating at 2000 rpm, and to be maintained at a surface temperature of 70° F, in air at 30° F. The lift coefficients for each of the sections were obtained from reference 25 (p. 664) in which tables of data are reproduced which were taken at the National Physical Laboratory. It was assumed that the angle of incidence of each of the blade elements was the same and equal to 6°. A plot of the results and a diagram of the propeller selected for the example are given in figures 5 and 6, respectively.

In calculating the heat loss from the surface of the propeller, two cases were postulated: (1) The flow remains laminar for the entire chord of the airfoil section for both the upper and lower surfaces, and (2) the flow is turbulent over the entire chord for the upper surface and laminar for the lower surface. In each of these cases an average f_c for each section was found by graphically integrating under the curves of figure 5 and the total heat loss was computed by adding the heat dissipation from each of the four sections. In the turbulent case this total heat loss was found to be 20,487 Btu per hour, whereas in the laminar case a value of 5772 Btu per hour was obtained.

Very little is known about the location of the transition point on propeller shapes, but it is probable that the flow is not completely laminar except possibly at positions close to the hub. Until further information is obtained regarding the nature of the flow about a propeller, it is probably safest to assume the turbulent value for the heat loss as a design estimate.

Department of Mechanical Engineering
University of California
Berkeley, Calif., September 29, 1944

APPENDIX A

METHOD OF DERIVATION AND PROCEDURE FOR CALCULATING POINT CONDUCTANCE

The general energy equation for an ideal gas may be written (reference 7, vol. II, p. 607)

$$\rho J C_p \frac{DT}{Dt} - \frac{Dp}{Dt} = Jk\nabla^2 T + \Phi \tag{92}$$

where D indicates a total derivative and Φ is the "dissipation function."

For the case of two-dimensional flow expression (92) reduces to the form

$$\rho J C_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) = Jk\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi \tag{93}$$

Under the usual postulates of boundary-layer theory the terms $\frac{\partial p}{\partial y}$ and $\frac{\partial^2 T}{\partial x^2}$ are set equal to zero; when $\mu\left(\frac{\partial u}{\partial y}\right)^2$ is substituted for Φ

$$\rho J C_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) = Jk\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{94}$$

If δ_H is the thickness of the thermal boundary layer, on integrating equation (94) from $y = 0$ to $y = \delta_H$,

$$JC_p \int_0^{\delta_H} \left(\rho\frac{\partial T}{\partial t} + \rho u\frac{\partial T}{\partial x} + \rho v\frac{\partial T}{\partial y}\right) dy - \int_0^{\delta_H} \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) dy = -Jk\int_0^{\delta_H} \frac{\partial^2 T}{\partial y^2} + \mu\int_0^{\delta_H} \left(\frac{\partial u}{\partial y}\right)^2 dy \tag{95}$$

since $\frac{\partial T}{\partial y}$ vanishes at $y = \delta_H$, which is the outer edge of the thermal layer. Now

$$\int_0^{\delta_H} \rho u \frac{\partial T}{\partial x} dy = \frac{\partial}{\partial x} \int_0^{\delta_H} \rho u T \, dy - \int_0^{\delta_H} T \frac{\partial}{\partial x} (\rho u) \, dy - (\rho u T)_{y=\delta_H} \frac{\partial \delta_H}{\partial x} \tag{96}$$

$$\int_0^{\delta_H} \rho v \frac{\partial T}{\partial y} dy = \rho v T \Big]_0^{\delta_H} - \int_0^{\delta_H} T \frac{\partial}{\partial y} (\rho v) \, dy \tag{97}$$

and

$$\int_0^{\delta_H} \rho \frac{\partial T}{\partial t} dy = \frac{\partial}{\partial t} \int_0^{\delta_H} \rho T \, dy - (\rho T)_{y=\delta_H} \frac{\partial \delta_H}{\partial t} - \int_0^{\delta_H} T \frac{\partial \rho}{\partial t} dy \tag{98}$$

Also v vanishes for $y = 0$; and for $y = \delta_H$ it follows from the equation of continuity that

$$(\rho v)_{y=\delta_H} = - \int_0^{\delta_H} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) \right] dy$$

$$= - \frac{\partial}{\partial t} \int_0^{\delta_H} \rho \, dy - \frac{\partial}{\partial x} \int_0^{\delta_H} \rho u \, dy + (\rho)_{y=\delta_H} \frac{\partial \delta_H}{\partial t} + (\rho u)_{y=\delta_H} \frac{\partial \delta_H}{\partial x} \tag{99}$$

Hence making use of continuity there is obtained

$$\int_0^{\delta_H}\left(\rho\frac{\partial T}{\partial t}+\rho u\frac{\partial T}{\partial x}+\rho v\frac{\partial T}{\partial y}\right)dy=\frac{\partial}{\partial t}\int_0^{\delta_H}\rho T\,dy$$

$$+\frac{\partial}{\partial x}\int_0^{\delta_H}\rho uT\,dy-T_1\left(\frac{\partial}{\partial t}\int_0^{\delta_H}\rho\,dy+\frac{\partial}{\partial x}\int_0^{\delta_H}\rho u\,dy\right)\tag{100}$$

where T_1 is the temperature at the outside of the layer. When equation (100) is substituted in equation (95), the following equation for the integrated energy balance on the thermal boundary layer is obtained:

$$JC_p\left[\frac{\partial}{\partial t}\int_0^{\delta_H}\rho T\,dy+\frac{\partial}{\partial x}\int_0^{\delta_H}\rho uT\,dy-T_1\left(\frac{\partial}{\partial t}\int_0^{\delta_H}\rho\,dy+\frac{\partial}{\partial x}\int_0^{\delta_H}\rho u\,dy\right)\right]$$

$$-\int_0^{\delta_H}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)dy=-Jk\left(\frac{\partial T}{\partial y}\right)_{y=0}+\mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)_{y=0}dy\tag{101}$$

The second aforementioned general integral relation is the well-known von Kármán "momentum equation,"

$$\frac{\partial}{\partial t}\int_0^{\delta}\rho u\,dy+\frac{\partial}{\partial x}\int_0^{\delta}\rho u^2\,dy-U\left(\frac{\partial}{\partial t}\int_0^{\delta}\rho\,dy+\frac{\partial}{\partial x}\int_0^{\delta}\rho u\,dy\right)=-\delta\frac{\partial p}{\partial x}-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}\tag{102}$$

If steady flow and an incompressible fluid are postulated and the dissipation term is neglected, these two equations reduce respectively to

$$\frac{\partial}{\partial x}\int_0^{\delta_H} uT\,dy - T_1\frac{\partial}{\partial x}\int_0^{\delta_H} u\,dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{103}$$

$$\frac{\partial}{\partial x}\int_0^{\delta} u^2\,dy - U\frac{\partial}{\partial x}\int_0^{\delta} u\,dy = -\frac{\delta}{\rho}\frac{\partial p}{\partial x} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{104}$$

When a fourth-power polynomial is postulated to represent the velocity distribution in the boundary layer, that is

$$\frac{u}{U} = ay + by^2 + cy^3 + dy^4 \tag{105}$$

with the boundary conditions

<u>at $y = \delta$:</u>

$$u = U$$

$$\frac{\partial u}{\partial x} = U'$$

$$\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 0$$

<u>at $y = 0$:</u>

$$u = 0$$

$$v = 0$$

therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho\nu}\frac{\partial p}{\partial x} = -\frac{UU'}{\nu} \tag{106}$$

Pohlhausen's expression (reference 27) is obtained

$$\frac{u}{U} = \left(2 + \frac{1}{6}\lambda\right)\left(\frac{y}{\delta}\right) + \left(-2 + \frac{1}{2}\lambda\right)\left(\frac{y}{\delta}\right)^3 + \left(1 - \frac{1}{6}\lambda\right)\left(\frac{y}{\delta}\right)^4 \tag{107}$$

where

$$\lambda = \frac{\delta^2}{\nu} \frac{dU}{dx} \tag{108}$$

Substitution of the drag at the wall from this polynomial into the momentum equation yields as an equation for λ

$$\lambda' = \frac{U'}{U} g(\lambda) + \frac{U''}{U'} \left[\lambda^2 h(\lambda) + \lambda\right] \tag{109}$$

where $g(\lambda)$ and $h(\lambda)$ are known tabulated functions of the argument (reference 7, vol I, p. 160), and primes denote differentiation with respect to x. At the point of stagnation $\lambda = 7.052$ and $\lambda' = 34.05 \frac{U'}{U}$; with these initial conditions equation (109) can be integrated either numerically or by the method of isoclines. If the temperature distribution is assumed to be of the same general form as that of velocity, namely

$$\frac{T}{T_1} = ay + by^2 + cy^3 + dy^4 \tag{110}$$

with the boundary conditions

$\underline{\text{at } y = \delta_H}$:

$$T = T_1$$

$$\frac{\partial T}{\partial x} = T'$$

$$\frac{\partial T}{\partial y} = 0$$

$$\frac{\partial^2 T}{\partial y^2} = 0$$

$\underline{\text{at } y = 0}$:

$$u = 0$$

$$v = 0$$

therefore

$$\frac{\partial^2 T}{\partial y^2} = 0 \tag{111}$$

there is obtained

$$\frac{T}{T_1} = 2\left(\frac{y}{\delta_H}\right) - 2\left(\frac{y}{\delta_H}\right)^3 + \left(\frac{y}{\delta_H}\right)^4 \tag{112}$$

Equation (103) may be written in the following form,

$$\frac{\partial}{\partial x} UT_1 \int_0^{\delta_H} \frac{u}{U} \frac{T}{T_1} dy - T_1 \frac{\partial}{\partial x} U \int_0^{\delta_H} \frac{u}{U} dy = - a\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{113}$$

If the "kinetic temperature rise" is set equal to zero, then the temperature at the edge of the boundary layer is the same as that in the free stream, that is $T_1 = T_\infty$, and

$$\frac{\partial}{\partial x}\left[U\left(\int_0^{\delta_H} \frac{u}{U} \frac{T}{T_1} dy - \int_0^{\delta_H} \frac{u}{U} dy\right)\right] = - \frac{a}{T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{114}$$

Substituting equations (107) and (112) and integrating,

$$\int_0^{\delta_H} \frac{u}{U} dy = \delta_H\left[(1 + 0.083\lambda)\left(\frac{\delta_H}{\delta}\right) + (-0.500 + 0.125\lambda)\left(\frac{\delta_H}{\delta}\right)^3 + (0.200 - 0.033\lambda)\left(\frac{\delta_H}{\delta}\right)^4\right] \tag{115}$$

$$\int_0^{\delta_H} \frac{u}{U} \frac{T}{T_1} dy = \delta_H\left[(0.866 + 0.072\lambda)\left(\frac{\delta_H}{\delta}\right) + (-0.479 + 0.119\lambda)\left(\frac{\delta_H}{\delta}\right)^3 + (0.194 - 0.032\lambda)\left(\frac{\delta_H}{\delta}\right)^4\right] \tag{116}$$

It is well known that as a good approximation (reference 7)

$$\frac{\delta_H}{\delta} \approx Pr^{-1/3} \tag{117}$$

so that we may define two functions

$$\int_0^{\delta_H} \frac{u}{U} \frac{T}{T_1} dy = F_1(\lambda, Pr) \equiv (0.866 + 0.072\lambda)Pr^{-1/3} + (-0.479 + 0.119\lambda)Pr^{-1} + (0.194 - 0.032\lambda)Pr^{-4/3} \tag{118}$$

$$\int_0^{\delta_H} \frac{u}{U} dy = F_2(\lambda, Pr) \equiv (1 + 0.083\lambda)Pr^{-1/3} + (-0.500 + 0.125\lambda)Pr^{-1} + (0.200 - 0.033\lambda)Pr^{-4/3} \tag{119}$$

and let

$$\varphi = F_1 - F_2$$

Then equation (113) reduces to

$$\frac{\partial}{\partial x} (U\delta_H\varphi) = -\frac{a}{T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{120}$$

Since all temperatures are measured with the surface temperature as a datum,

$$f_{c_x} = \gamma c_p \frac{\partial}{\partial x} (U\delta_H\varphi) \times 3600 \tag{121}$$

The procedure for calculating the point conductance in any given case would then be:

(1) Calculate U' and U'' either graphically or analytically.

(2) Compute the chordwise distribution of λ by Pohlhausen's method from the experimental pressure distribution. (See equation (57a).)

(3) Find δ by means of the relation $\lambda = \frac{\delta^2}{\nu} U'$.

(4) Find δ_H by means of the relation $\frac{\delta_H}{\delta} = Pr^{-1/3}$.

(5) Compute the functions F_1 and F_2 and hence φ.

(6) Substitute in equation (121) and evaluate the derivative graphically.

It is seen that this procedure permits extension to cases in which compressibility and dissipation effects are to be considered. When $\rho = \frac{p}{RT}$ and $\frac{\partial p}{\partial x} = -\rho UU'$ are substituted in the general energy equation for steady flow,

$$JC_p\left(\frac{\partial}{\partial x}\int_0^{\delta_H}\rho uT\,dy - T_1\frac{\partial}{\partial x}\int_0^{\delta_H}\rho u\,dy\right) - \int_0^{\delta_H} u\,\frac{\partial p}{\partial x}\,dy = -Jk\left(\frac{\partial T}{\partial y}\right)_{y=0} + \mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (122)$$

then

$$JC_p\left(\frac{1}{R}\frac{\partial}{\partial x}\int_0^{\delta_H} pu\,dy - T_1\frac{\partial}{\partial x}\int_0^{\delta_H}\frac{pU}{RT}\,dy\right) + \frac{pUU'}{R}\int_0^{\delta_H}\frac{u}{T}\,dy = -Jk\left(\frac{\partial T}{\partial y}\right)_{y=0} + \mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 \qquad (123)$$

$$JC_p\left[\frac{1}{R}\frac{\partial}{\partial x}\left(p\int_0^{\delta_H} u\,dy\right) - \frac{T_1}{R}\frac{\partial}{\partial x}\left(p\int_0^{\delta_H}\frac{u}{T}\,dy\right)\right]$$

$$+ \frac{pUU'}{R}\int_0^{\delta_H}\frac{u}{T}\,dy = -Jk\left(\frac{\partial T}{\partial y}\right)_{y=0} + \mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (124)$$

$$JC_p\left(\frac{1}{R}\frac{\partial p}{\partial x}\int_0^{\delta_H} u\,dy + \frac{p}{R}\frac{\partial}{\partial x}\int_0^{\delta_H} u\,dy - \frac{T_1}{R}\frac{\partial p}{\partial x}\int_0^{\delta_H}\frac{u}{T}\,dy - \frac{pT_1}{R}\frac{\partial}{\partial x}\int_0^{\delta_H}\frac{u}{T}\,dy\right)$$

$$+ \frac{pUU'}{R}\int_0^{\delta_H}\frac{u}{T}\,dy = -Jk\left(\frac{\partial T}{\partial y}\right)_{y=0} + \mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (125)$$

$$JC_p\left[\frac{1}{R}\,\rho U'\int_0^{\delta_H}\frac{u}{U}\,dy+\frac{p}{R}\frac{\partial}{\partial x}\left(U\int_0^{\delta_H}\frac{u}{U}\,dy\right)+\frac{1}{R}\,\rho U^2U'\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy-\frac{pT_1}{R}\frac{\partial}{\partial x}\left(\frac{U}{T_1}\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy\right)\right]$$

$$+\frac{pU'}{RT_1}\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy=-Jk\left(\frac{\partial T}{\partial y}\right)_{y=0}+\mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 \tag{126}$$

$$JC_p\left(\frac{pU'}{R^2T_1}\int_0^{\delta_H}\frac{u}{U}\,dy+\frac{pU'}{R}\int_0^{\delta_H}\frac{u}{U}\,dy+\frac{pU}{R}\frac{\partial}{\partial x}\int_0^{\delta_H}\frac{u}{U}\,dy+\frac{pU^2U'}{R^2T_1}\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy-\frac{pU'}{R}\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy\right)$$

$$+\frac{pUT_1'}{RT_1}\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy-\frac{pU}{R}\frac{\partial}{\partial x}\int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy=-Jk\left(\frac{\partial T}{\partial y}\right)_{y=0}+\mu\int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 dy \tag{127}$$

The polynomials of equations (107) and (112) permit the evaluation of the integrals

$$\int_0^{\delta_H}\frac{u}{U}\,dy,\quad \int_0^{\delta_H}\frac{u}{U}\frac{T_1}{T}\,dy,\quad\text{and}\quad \int_0^{\delta_H}\left(\frac{\partial u}{\partial y}\right)^2 dy \tag{128}$$

When these integrals are called $F_1(\lambda_1 Pr)$, $F_2(\lambda_1 Pr)$, and $F_3(\lambda_1 Pr)$, respectively, equation (127) becomes

$$\frac{JC_p p}{R}\left[\left(\frac{U}{RT_1}+U'\right)\delta_H F_1+\left(\frac{U^2U'}{RT_1}-U'-\frac{U'T_1'}{T_1}+\frac{U'}{JC_pT}\right)\delta_H F_2+U\left(\delta_H F_1'+\delta_H' F_1\right)\right.$$

$$\left.-UT_1{}^2\left(\delta_H' F_2+\delta_H F_2'\right)\right]=-Jk\left(\frac{\partial T}{\partial y}\right)_{y=0}+\mu F_3 \tag{129}$$

or

$$f_{c_x} = -\frac{C_p p}{R}\left[\frac{U'(1 + RT_1)}{RT_1}\,\delta_H F_1 + \left(\frac{U^2 U'}{RT_1} - U' - \frac{U' T_1'}{T_1} + \frac{U'}{JC_p T}\right)\delta_H F_2\right.$$

$$\left. + U\left(\delta_H' F_1 + \delta_H F_1'\right) - U T_1{}^2\left(\delta_H F_2 + \delta_H F_2'\right)\right] + \mu F_3 \qquad (130)$$

APPENDIX B

DISCUSSION OF CERTAIN BOUNDARY-LAYER CONCEPTS

In the development of boundary-layer theory, certain basic postulates are tacitly made which, because they are not clearly stated, may cause the engineer who is not primarily an aerodynamicist considerable difficulty. The purpose of this appendix is to discuss several of these points somewhat more thoroughly than is done in the usual references on aerodynamics.

There are two general methods of analyzing boundary-layer problems:

(a) The solution of the boundary-layer differential equations (equation (132)).

(b) The solution of the boundary-layer momentum equation (equation (104)).

The first part of the appendix deals with method (a); the second part deals with method (b).

Boundary-Layer Differential Equations

The incompressible-flow boundary-layer differential equations are derived from the general hydrodynamic equations with the following postulates:

(a) The flow pattern about the object is two-dimensional.

(b) The thickness of the region next to the solid surface, in which the velocity gradient is large, is small compared with the other linear dimensions of the object.

(c) The flow is incompressible.

(d) There is no separation of the flow from the solid object.

(e) The fluid in contact with the solid surface has no velocity relative to that surface.

Reference 7 (vol. II, p. 610) presents a derivation of the boundary-layer equations, for curved surfaces, on the basis of these postulates. The boundary-layer equations are: (See also fig. 1.)

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\
&-ku^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}\right\} \tag{131}$$

The following further postulates are usually made in solving equation (131):

(e) Velocity u is not a function of time.

(f) $\int_0^\infty \left(\frac{\partial p}{\partial y}\right) dy$ is negligibly small even along a curved surface, so that the pressure is postulated to be invariable in the direction normal to the surface.

The equations then reduce to:

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}\right\} \tag{132}$$

By making postulate (f), all problems of two-dimensional boundary-layer theory become problems of flow along a plane surface (flat plate) along which the pressure gradient $\frac{\partial p}{\partial x}$ varies in some given manner.

The boundary conditions imposed on the boundary-layer equations, regardless of the variation of $\frac{\partial p}{\partial x}$ are

$$\left.\begin{array}{ll}\text{at} & y = 0 \\ \text{and} & x = \infty\end{array}\right\}\begin{cases} u = 0 \\ v = 0 \end{cases}$$

$$\begin{array}{ll}\text{at} & y = \infty \\ \text{and} & x = 0\end{array}\begin{cases} u = U \\ \dfrac{\partial u}{\partial y} = 0 \\ \dfrac{\partial^2 u}{\partial y^2} = 0 \end{cases} \tag{133}$$

The extension of the solution to $y = \infty$ assumes that the whole fluid field is in viscous motion. With these boundary conditions, the velocity profile at any x will have the form shown in figure 8.

Since at $y = \infty$, $u = U$, $\frac{\partial u}{\partial y} = 0$, and $\frac{\partial^2 u}{\partial y^2} = 0$, the boundary-layer equations far from the plate become:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{\partial U}{\partial x} \tag{134}$$

Integration of equation (134) yields Bernoulli's equation.

Since $\frac{\partial p}{\partial x}$ does not vary with y, the boundary-layer equations become:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \tag{135}$$

As was previously mentioned, these equations are strictly applicable to a flat plate only. Approximate solutions of these equations over curved shapes are obtained by postulating various variations of U with x and substituting this resulting form of $U\frac{\partial U}{\partial x}$ into the foregoing equations. Only certain special forms of $U = f_{(x)}$ can be handled mathematically.

The following table was compiled from the solutions of the boundary-layer equations presented in reference 7, and reveals how flow about various objects is identified with certain variations of U along a plane surface.

U	$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{\partial U}{\partial x}$	"Equivalent flow system"
Constant	0	Flow along flat plate
$\beta_1 x$	$\beta_1{}^2 x$	Flow near stagnation point of cylinder
$\frac{c}{x}$	$-\frac{c^2}{x^3}$	Flow in converging channel
cx^m	$c^2 m\, x^{2m-1}$	$m = 0$ Flat plate $m = 1$ Stagnation point $m = -1$ Converging channel
$\sum_{m=1}^{m=\infty} cx^m$	$\sum_{m=1}^{m=\infty} c^2 m\, x^{2m-1}$	Any curved shape if series is known from experimental data

A diagram of the velocity distribution near the stagnation point of a cylinder, obtained from a solution of the boundary-layer equations, is shown in figure 9. A discrepancy with the actual physical system is immediately apparent. Figure 9 shows the velocity far from the object increasing linearly with x, whereas obviously in the physical system the velocity far from the surface remains constant.

The flow pattern about the actual physical system is shown in figure 10. This figure reveals that the velocity in the physical system, at any fixed value of x, increases as y increases, reaches a maximum, and then decreases to an asymptotic magnitude u_∞ as $y \rightarrow \infty$. Comparison of the velocity distribution about a cylinder obtained from potential theory (zero viscosity) with the experimental points would show a good check until the surface of the cylinder was approached closely; the potential solutions would then become greatly in error. Conversely the solution of the boundary-layer equations would show good agreement with the data for magnitudes of y from zero to about the point of maximum velocity; beyond this point the boundary-layer solution would deviate greatly from the data.

Because of the deviation of the boundary-layer solution from experimental results for flow around curved objects, a limit must be placed on the region in which the boundary-layer solution is applicable. This limit is called the thickness of the boundary layer. In analyzing boundary-layer problems aerodynamicists as a rule concern themselves with what occurs within this boundary layer and usually neglect completely the flow outside of this boundary layer.

It is evident that a unique definition of the thickness of the boundary layer δ is difficult to establish. Several possible definitions are:

(a) The point y, where the boundary layer and potential solutions intersect

(b) The point y, where $u = 0.99U$, or any other arbitrary fraction of U

(c) The point y at which the total pressure reaches a fraction F of the total pressure in the free stream

Definition (c) of δ (for incompressible flow) is equivalent to the point y at which $u = \sqrt{F}\,U$.

(d) A characteristic length, called the displacement thickness δ_1 may be defined, so that

$$\int_0^\infty U\,dy - (U\delta_1) = \int_0^\infty u\,dy \qquad (136)$$

If the flow along the body obeyed the boundary-layer equation, the rate of flow of fluid across any x would be $\int_0^\infty u\,dy$. This is less than the quantity $\int_0^\infty U\,dy$ because of the retardation of the flow near the surface of the object. The difference is called $U\delta_1$, thus defining the characteristic length δ_1. The length δ_1 is not the boundary-layer thickness, in itself, but is related to it.

The exact definition of boundary-layer thickness adopted should not influence the final results of a boundary-layer analysis as long as the definition is utilized in a consistent manner, since the analytical solutions of the boundary-layer equations never involve the boundary-layer thickness directly.

From figure 10 it is evident that in the physical system the velocity outside the boundary layer varies with y. In aerodynamic analysis, however, the velocity U is called the velocity at the edge of the boundary layer. The velocity U is defined by

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \qquad (134a)$$

whence

$$\frac{U^2}{2} = -\frac{1}{\rho} p + \text{Constant}$$

$$p + \frac{\rho U^2}{2} = \text{Constant}$$

Thus U may be calculated from the pressure distribution around a curved object. It is defined by equation (134a) and is the velocity just outside the boundary layer only in the sense that for any fixed value of x, at one point y outside the boundary layer, the velocity U will exist. The correct value for the velocity at the edge of the boundary layer will, of course, depend upon the definition of the boundary layer adopted. Utilizing U indiscriminately for the velocity at the edge of the boundary layer in conjunction with different definitions of the boundary-layer thickness will lead to erroneous results.

It has been shown that the boundary-layer solution is not applicable outside the boundary layer. If now it is postulated that the flow outside the boundary layer does not affect the velocity inside the boundary layer, the laminar-boundary-layer equations may be applied to a system even though the flow outside the boundary layer is turbulent, as long as the flow within the boundary layer remains laminar. Once the flow within the boundary layer becomes even partly turbulent, the boundary-layer equations are no longer applicable.

Momentum Equations

By the considerations of the flow within the boundary layer it may be established that a certain thickness δ exists in which the boundary-layer equations are applicable. If it is further postulated that: (a) the exact form of the velocity distribution outside the boundary layer is of no importance whatsoever in determining the behavior inside the boundary layer and (b) that a velocity U calculated from $\rho \frac{U^2}{2} + p = \text{Constant}$ exists at the edge of the boundary layer, certain simplified equations may be written which allow the approximate analysis of the velocity distribution within the boundary layer.

Since the two postulates mentioned are not exactly true, the momentum equations are only approximations. Experimental results indicate that the approximation is fairly good. The momentum equation for steady flow (see appendix A) is

$$\frac{\partial}{\partial x} \int_0^\delta u^2 \, dy - U \frac{\partial}{\partial x} \int_0^\delta u \, dy = -\frac{\delta}{\rho} \frac{\partial p}{\partial x} - \nu \left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (104)$$

It should be particularly noted that integrations extend only to δ instead of ∞ as was the case for the boundary-layer equations. Thus the boundary-layer thickness is of primary importance in the momentum equations, although its definition was really not essential in the analytical solution of the boundary-layer equations.

The momentum equations are solved by expressing the velocity ratio $\frac{u}{U}$ as a function of $\frac{y}{\delta}$. A typical expression for $\frac{u}{U}$ (appendix A) is

$$\frac{u}{U} = a\left(\frac{y}{\delta}\right) + b\left(\frac{y}{\delta}\right)^2 + c\left(\frac{y}{\delta}\right)^3 + d\left(\frac{y}{\delta}\right)^4$$

Since only the region between $\frac{y}{\delta} = 0$ and $\frac{y}{\delta} = 1$ is being considered, the following approximate boundary conditions may be imposed on $\frac{u}{U}$

$$\left.\begin{array}{llll}
\frac{\partial u}{y} = 0 & \text{at } \frac{y}{\delta} = 1 & \frac{u}{U} & \text{at } \frac{y}{\delta} = 0 \\
\frac{\partial^2 u}{y^2} = 0 & \text{at } \frac{y}{\delta} = 1 & \frac{\partial u}{\partial y} \text{ is finite} & \text{at } \frac{y}{\delta} = 0 \\
\frac{\partial^3 u}{y^3} = 0 & \text{at } \frac{y}{\delta} = 1 & \frac{\partial^2 u}{\partial y^2} = \frac{P}{x} & \text{at } \frac{y}{\delta} = 0 \\
 & & \frac{\partial^3 u}{\partial y^3} = 0 & \text{at } \frac{y}{\delta} = 0
\end{array}\right\} \qquad (137)$$

(See reference 7, vol. I, p. 156). These boundary conditions, which are not exactly true in the physical system, allow, however, the evaluation of the constants a, b, c, d, and so forth. Obviously the velocity distribution is only applicable between $\frac{y}{\delta} = 0$ and $\frac{y}{\delta} = 1$. Outside $\frac{y}{\delta} = 1$ there is no relation between the polynomial expressing $\frac{u}{U}$ and the experimental velocity distribution.

Having $\frac{u}{U}$ as a function of $\frac{y}{\delta}$, the momentum equation may be solved for the shear at the wall, the boundary-layer thickness δ, and the displacement thickness δ_1. In addition, for purposes of analysis the momentum thickness defined by

$$\theta U^2 = \int_0^\delta (U - u)u \, dy \qquad (138)$$

is often calculated by the momentum equation method.

The advantages of the momentum method are the relative simplicity of accounting for variable pressure gradients as a function of x and the fact that it may be applied approximately to both laminar and turbulent boundary layers.

APPENDIX C

SOME REMARKS ON THE SIMILARITY OF VELOCITY AND TEMPERATURE DISTRIBUTIONS IN THE PRESENCE OF A PRESSURE GRADIENT

As was discussed in the section of the text LAMINAR REGIME, Method of Squire, inspection of the boundary-layer equations, for both laminar and turbulent boundary layers, reveals that within the boundary layer the pressure gradient will probably cause only secondary differences in the temperature distribution as compared with the velocity distribution. This conclusion is even more important for the region outside the boundary layer. As shown in figure 7 and as discussed in appendix B, the solution of the boundary-layer equations, even in the presence of a pressure gradient, yields values of U and T which asymptotically approach certain magnitudes of U and T which are supposed to exist far from the solid-fluid interface.

Physically, however, in the presence of a pressure gradient the actual velocity outside of the boundary layer does not approach the velocity U asymptotically, but (in a region of negative pressure gradient) reaches a maximum and then decreases to a magnitude of the free-stream velocity u_∞. The temperature distribution, however, (neglecting frictional heating) approaches T_∞ asymptotically.

It is apparent therefore that, at least outside the boundary layer, the pressure gradient affects the velocity distribution much more than it affects the temperature distribution.

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TABLE I

CALCULATIONS FOR LAMINAR BOUNDARY LAYER FOR JOUKOWSKI AIRFOIL

[c = 7.78 ft, u_∞ = 253 ft/sec, γ = 0.078 lb/ft^3,
C_p = 0.24 Btu/(lb)(°F), Pr = 0.72, α = 1.5°]

x/c	$\frac{U}{u_\infty}$	Re_x	$\frac{0.332}{\sqrt{Re_x}}$	$[A]^{-1/2}$ (1)	$[B]^{-1/2}$ (2)	$Pr^{-2/3}$	$\left(\frac{\delta_1}{\delta_2}\frac{1}{Pr}\right)$
0.06	1.270	76.4×10^4	3.80×10^{-4}	1.197	1.059	1.242	1.22
.10	1.276	131.5	2.81	1.121	1.050	1.242	1.24
.16	1.276	209	2.22	1.070	1.012	1.242	1.24
.20	1.272	263	1.99	1.042	1.012	1.242	1.24
.28	1.264	366	1.68	1.003	.997	1.242	1.25
.36	1.251	466	1.49	.968	.980	1.242	1.26
.44	1.229	559	1.37	.910	.959	1.242	1.28
.52	1.218	655	1.26	.895	.932	1.242	1.30
.60	1.198	744	1.18	.845	.895	1.242	1.32
.72	1.162	860	1.14	.770	.848	1.242	1.36

x/c	f_{c_x} as determined by -			
	Allen and Look (equation (23)) (3)	Frick and McCullough (equation (28))	Martinelli and others (equation (33))	Squire (equation (39))
0.06	9.8	12.2	10.2	10.6
.10	6.8	8.50	7.59	7.95
.16	5.16	6.42	5.98	6.05
.20	4.51	5.60	5.38	5.43
.28	3.65	4.54	4.51	4.54
.36	3.08	3.83	3.94	3.96
.44	2.60	3.24	3.57	3.51
.52	2.35	2.94	3.27	3.18
.60	2.04	2.53	3.00	2.86
.72	1.74	2.16	2.82	2.61

NACA

1
$$A = \left[\frac{\int_0^{\frac{x}{c}} \left(\frac{u}{U}\right)^{8.17} d\left(\frac{x}{c}\right)}{\left(\frac{u}{U}\right)^{8.17}\left(\frac{x}{c}\right)}\right]$$

2
$$B = \left(\frac{\int_0^x U^5\, dx}{xU^5}\right)$$

3 Calculations based on Pr value of 1.

TABLE II

CALCULATIONS FOR TURBULENT BOUNDARY LAYER FOR NACA 65,2-016 AIRFOIL

[ζ calculated in reference 11; $\frac{U'}{U}\left(\frac{\delta}{\theta}+1\right)\theta$ approximately calculated by utilizing $\frac{\delta}{\theta} = 1.28$]

x/c	$\frac{U}{u_\infty}$	Re_x	$\frac{0.0296}{Re_x^{0.2}}$	$\frac{Re_x^{0.2}}{0.0296\zeta^2}$	$\frac{U'}{U}\left(\frac{\delta}{\theta}+1\right)\theta$	A (1)	$Pr^{-2/3}$
0.01	1.61	14.0×10^4	0.00277	1.092	-2.2×10^{-4}	1.17	1.24
.03	1.54	40.2	.00224	.866	-3.74	1.13	1.24
.05	1.50	65.2	.00204	.890	-3.54	1.12	1.24
.09	1.44	11.3	.00182	.901	-2.94	1.12	1.24
.11	1.42	13.6	.00174	.938	-1.81	1.12	1.24
.21	1.38	25.2	.00155	.948	-1.71	1.11	1.24
.31	1.35	36.5	.00144	.963	-1.09	1.11	1.24
.41	1.35	47.9	.00136	.982	-2.76	1.11	1.24
.51	1.32	58.4	.00131	.980		1.11	1.24

x/c	f_{c_x} as determined by -				
	Martinelli and others (equation (71))	Frick and McCullough I (equation (70))	Frick and McCullough II (equation (73))	Squire I (equation (75))	Squire II (equation (81))
0.01	78.1	68.5	84.4	80.4	73.4
.03	60.5	42.4	52.6	48.4	50.6
.05	53.7	38.5	47.8	43.6	46.0
.09	45.8	33.2	41.3	37.7	39.2
.11	43.6	32.4	40.0	36.4	36.0
.21	37.6	28.8	35.6	32.3	32.2
.31	34.3	26.7	33.3	29.9	28.8
.41	32.2	25.4	31.6	28.4	30.8
.51	29.7	23.5	29.1	26.3	

1.
$$A = \frac{1}{1 + \frac{5}{\zeta}\left\{(Pr - 1) + \log_e\left[1 + \frac{5}{6}(Pr - 1)\right]\right\}}$$

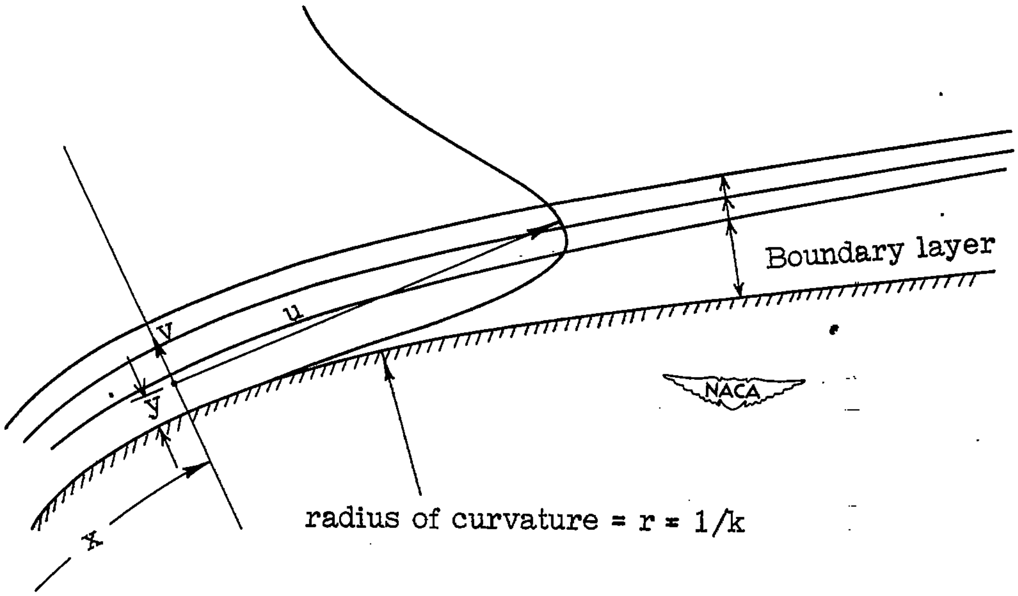

Figure 1.- Typical boundary layer.

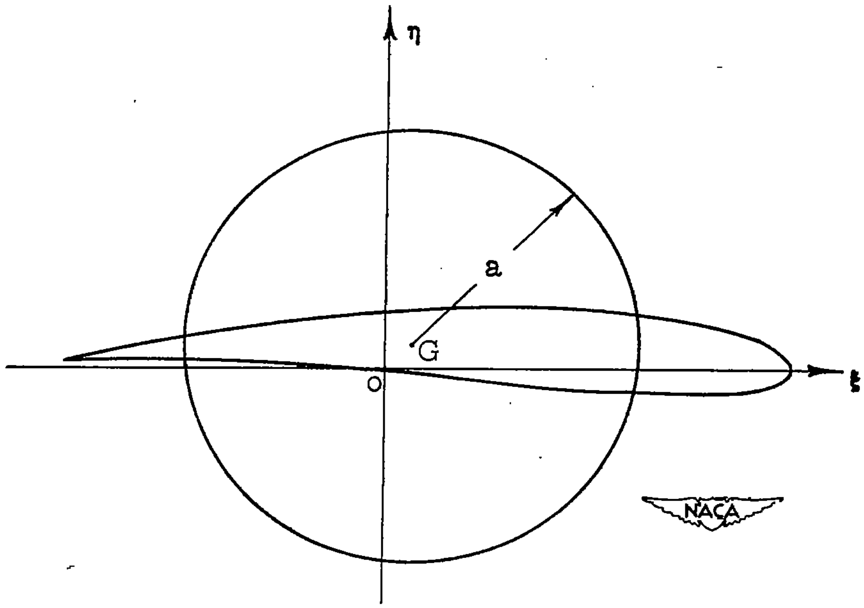

Figure 2.- Joukowski profile used in comparison of methods for laminar regime.

$$\xi/a = (\cos\theta + 0.04)\left[1 + \frac{0.83}{(\cos\theta + 0.04)^2 + (\sin\theta + 0.05)^2}\right]$$

$$\eta/a = (\sin\theta + 0.05)\left[1 - \frac{0.83}{(\cos\theta + 0.04)^2 + (\sin\theta + 0.05)^2}\right]$$

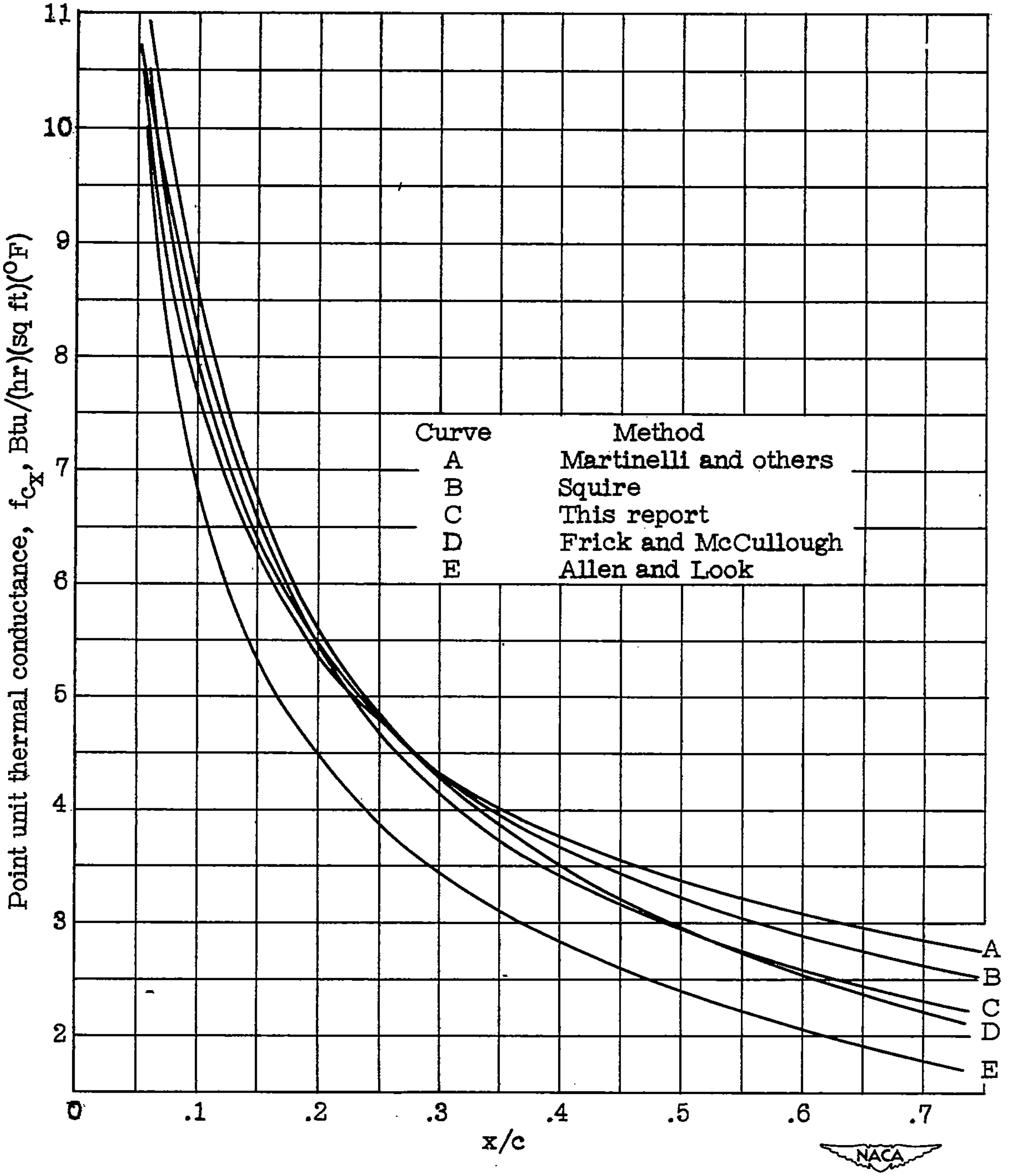

Figure 3.- Comparison of methods for calculating point unit thermal conductance in laminar regime for an airfoil section (Joukowski profile). True airspeed, 253 feet per second; air temperature, 30° F; wing temperature, 70° F; angle of attack, 1.5°; wing chord, 7.78 feet.

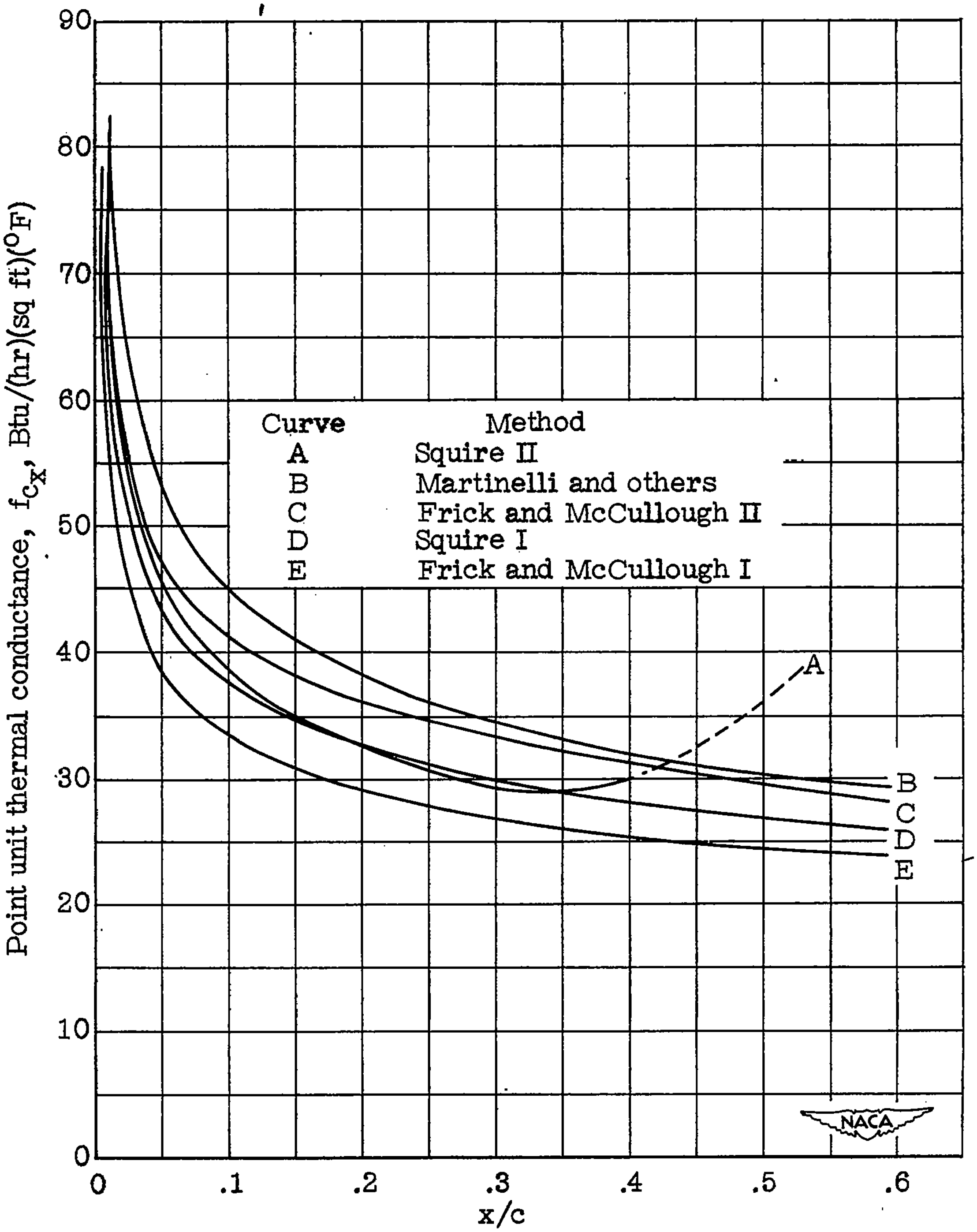

Figure 4.- Comparison of methods for calculating point unit thermal conductance in turbulent regime for airfoil section (NACA 65,2-016). True airspeed, 206 feet per second; air temperature, 30° F; wing temperature, 70° F; lift coefficient, 0.55; wing chord, 7.0 feet.

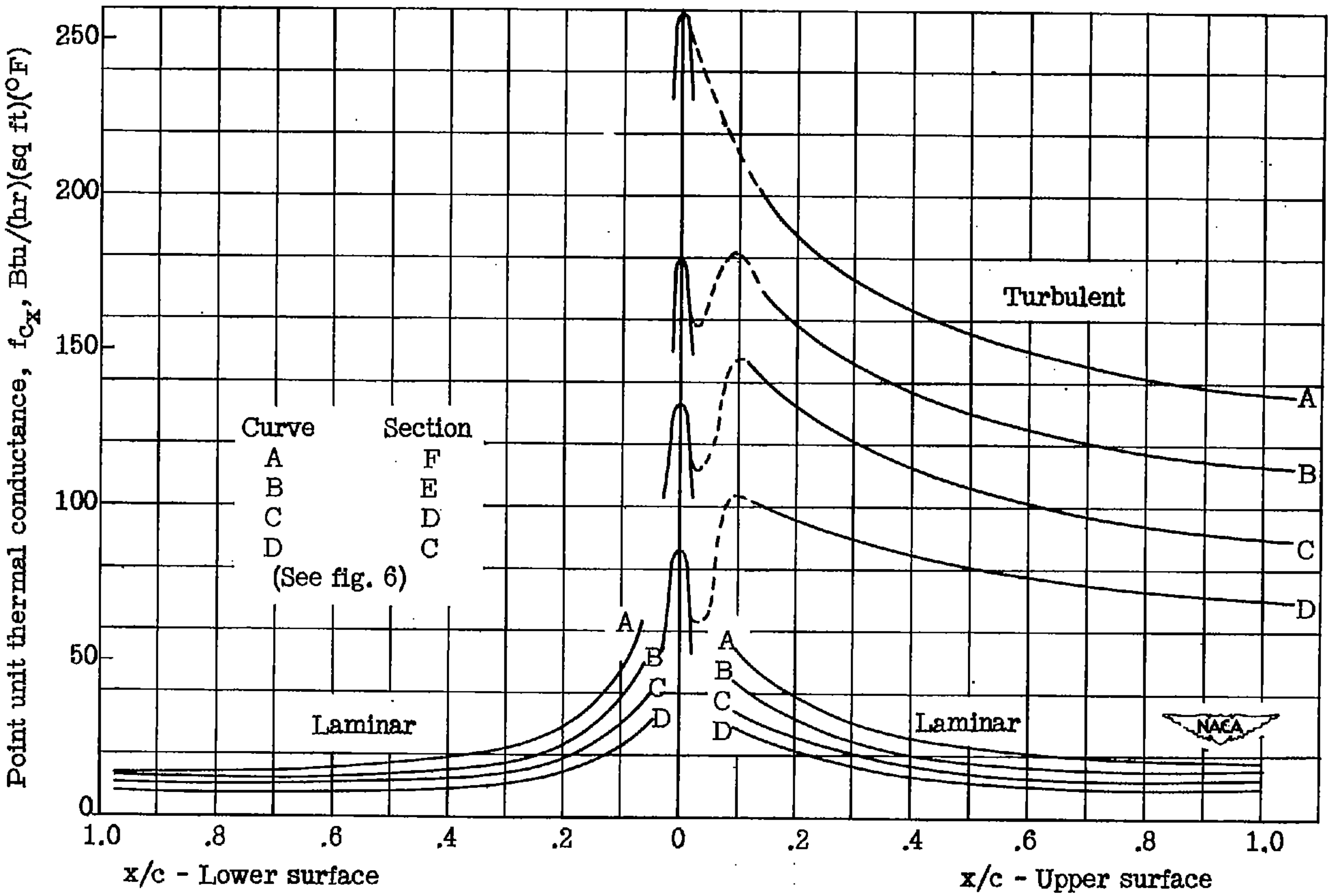

Figure 5.- Point unit thermal conductances on lower and upper surfaces of a propeller. Radius, 5 feet; speed, 2000 rpm; angle of attack, 6°; surface temperature, 70° F; air temperature, 30° F.

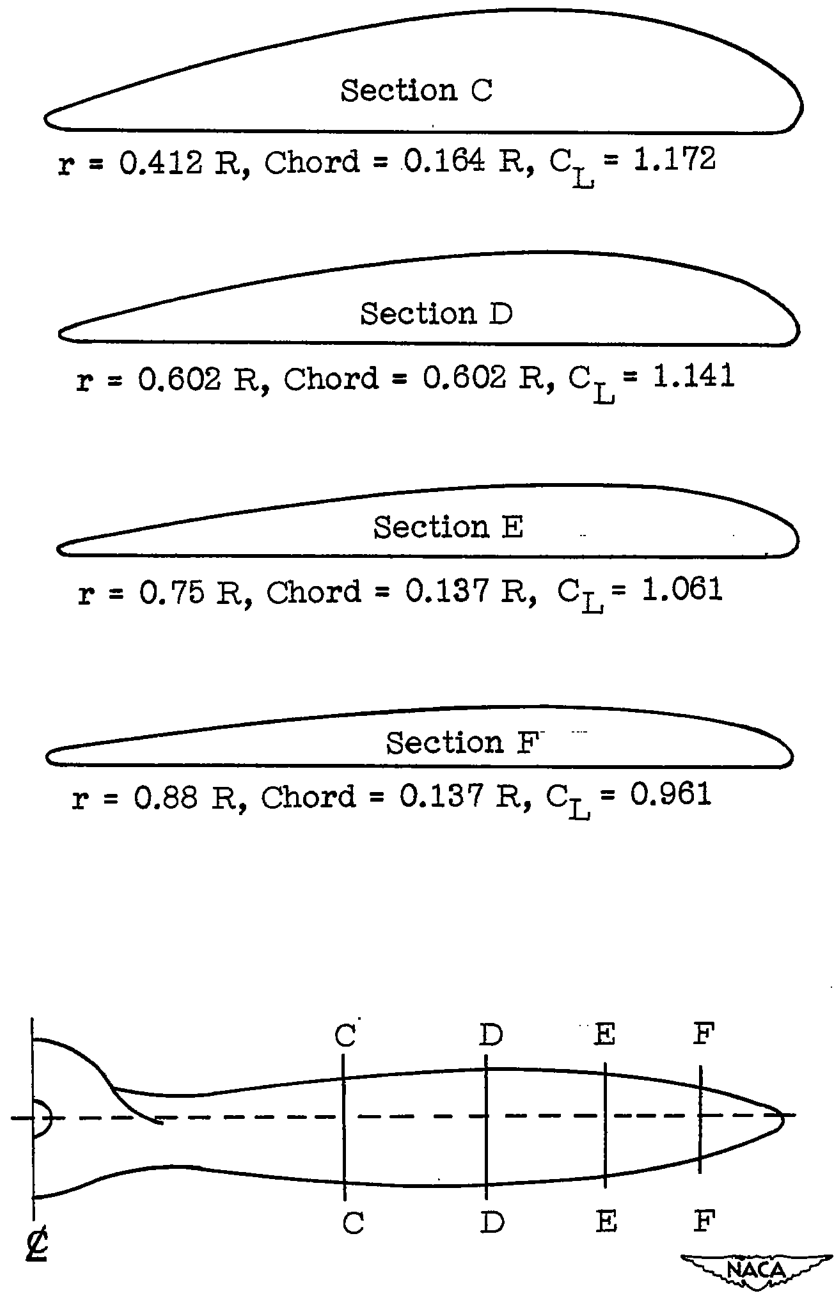

Figure 6.- Diagram of propeller section used in illustrative example. Angle of attack, 6°.

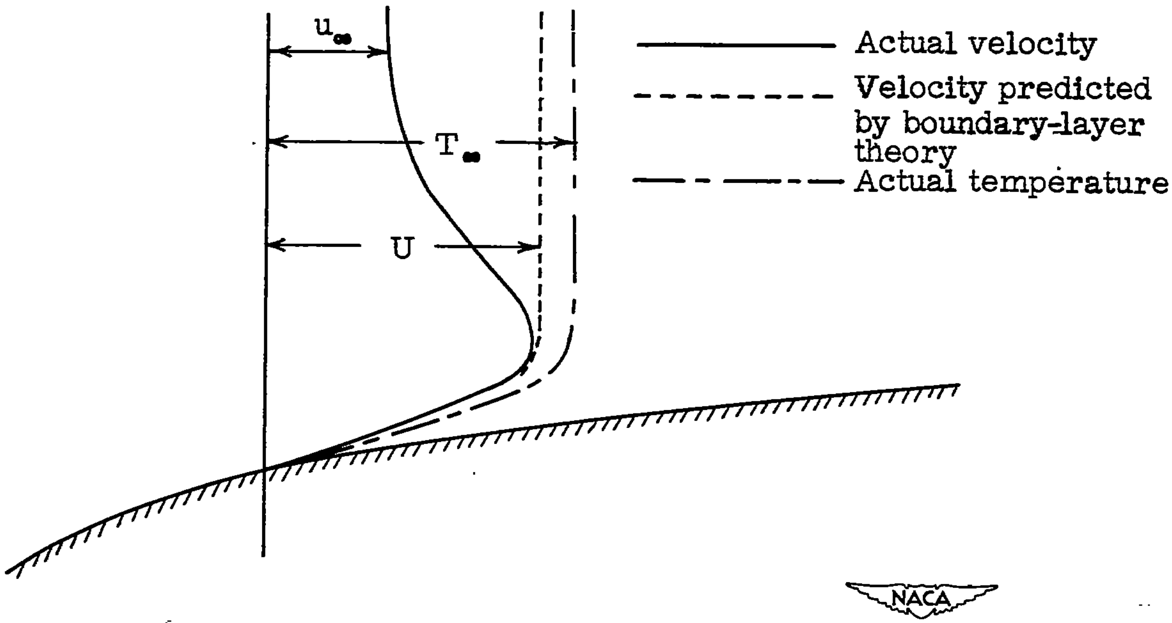

Figure 7.- Velocity and temperature distributions in presence of pressure gradient.

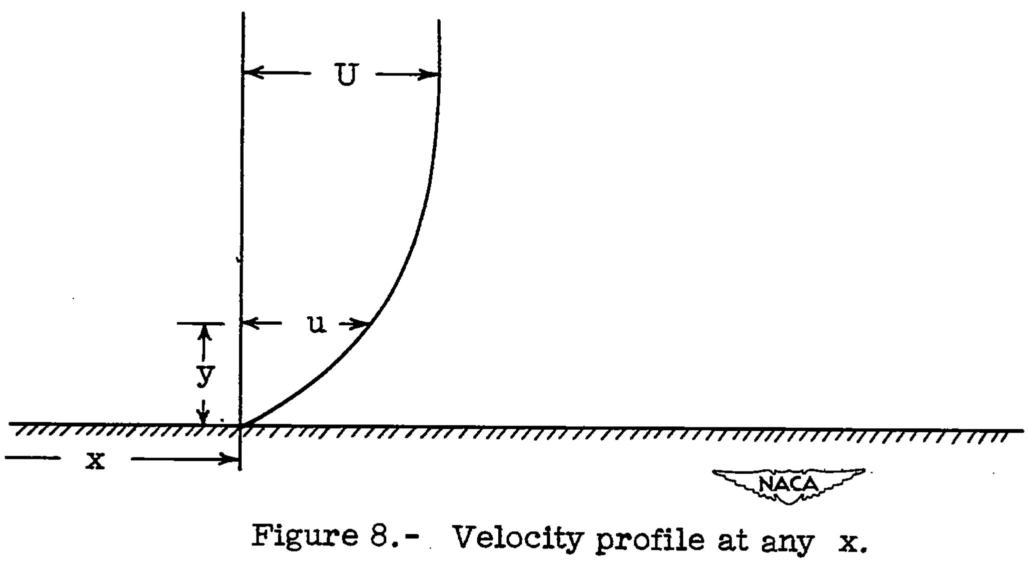

Figure 8.- Velocity profile at any x.

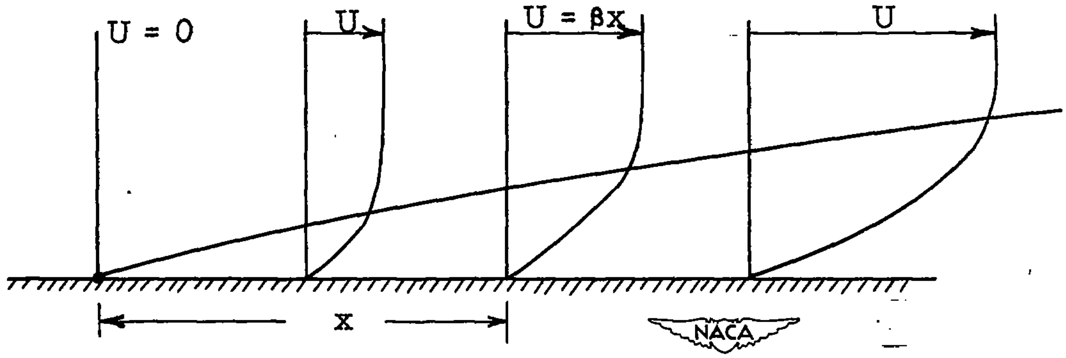

Figure 9.- Diagram of velocity distribution near stagnation point of cylinder obtained from solution of boundary-layer equation.

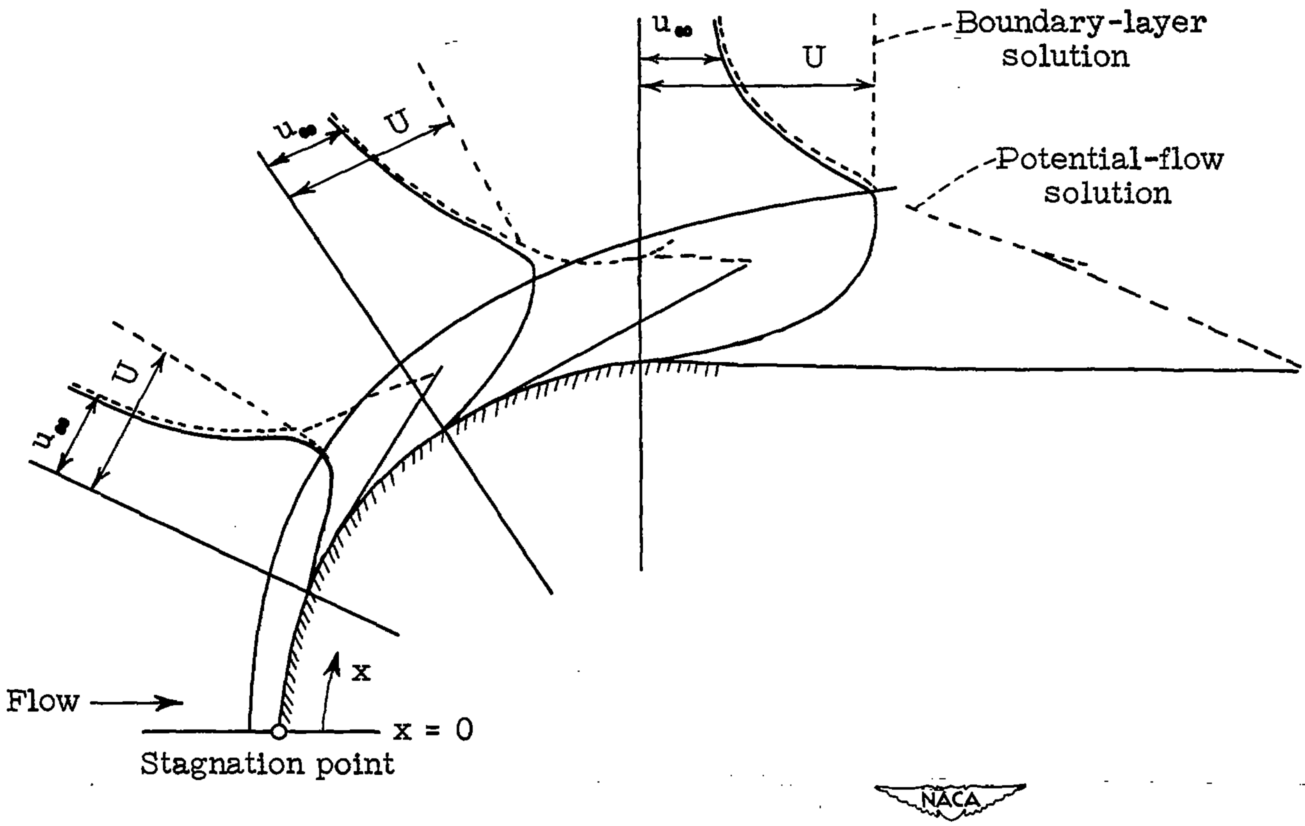

Figure 10.- Flow pattern about actual physical system.